Tree-based tensor formats

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Abstract The main goal of this paper is to study the topological properties of tensors in tree-based Tucker format. These formats include the Tucker format and the Hierarchical Tucker format. A property of the so-called minimal subspaces is used for obtaining a representation of tensors with either bounded or fixed tree-based rank in the underlying algebraic tensor space. We provide a new characterisation of minimal subspaces which extends the existing characterisations. We also introduce a definition of topological tensor spaces in tree-based format, with the introduction of a norm at each vertex of the tree, and prove the existence of best approximations from sets of tensors with bounded tree-based rank, under some assumptions on the norms weaker than in the existing results.

Keywords Tensor spaces \cdot Tree-based tensor format \cdot Tree-based rank \cdot Best approximation

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1 Introduction

Tensor approximation methods play a central role in the numerical solution of high dimensional problems arising in a wide range of applications. The reader is referred to the monograph [9] and surveys [11,5,13,1] for an introduction to tensor numerical methods and an overview of recent developments in the field. Low-rank tensor formats based on subspaces are widely used for complexity reduction in the representation of high-order tensors. Two of the most popular formats are the Tucker format and the Hierarchical Tucker format [8] (HT for short). It is possible to show that the Tensor Train format [14], introduced originally by Vidal [15], is a particular case of the HT format (see e.g. Chapter 12 in [9]). In the framework of topological tensor spaces, first results have been obtained on the existence of a best approximation in each fixed set of tensors with bounded rank [3]. In particular, this allows to construct, on a theoretical level, iterative methods for nonlinear convex optimisation problems over reflexive tensor Banach spaces [4]. More generally, this is a crucial property for proving the stability of algorithms using tree-based tensor formats.

The Tucker and the HT formats are completely characterised by a rooted tree together with a finite sequence of natural numbers associated to each vertex of the tree, denominated the tree-based rank. Each number in the treebased rank is associated with a class of subspaces of fixed dimension. It can be shown that for a given tree, every element in the tensor space possesses a unique tree-based rank. In consequence, given a tree, a tensor space is a union of sets indexed by the tree-based ranks. It allows to consider for a given tree two kinds of sets in a tensor space: the set of tensors of fixed tree-based rank and the set of tensors of bounded tree-based rank.

This paper provides new results on the representation of tensors in general tree-based Tucker formats, in particular on a characterisation of minimal subspaces compatible with a given tree. It also provides a definition of topological tensor spaces associated with a given tree, and provides new results on the existence of best approximations from sets of tensors with bounded tree-based rank.

The paper is organised as follows. In Section 2 , we introduce the treebased tensors as a generalisation, at algebraic level, of the hierarchical tensor format. Moreover, we provide a new characterisation of the minimal subspaces of tree-based tensors extending the previous results obtained in [3], and introduce the definition of tree-based rank. Another main result of this section is Theorem 1, which provides a characterisation for the representation for the set of tensors with fixed tree-based rank. In Section 3 we introduce a definition of topological tensor spaces in tree-based format, with the introduction of a norm at each vertex of the tree. Finally in Section 3, we prove the existence of best approximations from sets of tensors with bounded tree-based rank under some assumptions on the norms that are weaker than the ones introduced in [3].

2 Algebraic tensors in the tree-based format

2.1 Preliminary definitions and notations

Let $D = \{1, 2, ..., d\}$ be a finite index set, and let V_j $(1 \le j \le d)$, be vector spaces. Concerning the definition of the algebraic tensor space

$$\mathbf{V}_D := {}_a \bigotimes_{j=1}^d V_j \;,$$

we refer to Greub [6]. As underlying field we choose \mathbb{R} , but the results hold also for \mathbb{C} . The suffix 'a' in $_{a} \otimes_{j=1}^{d} V_{j}$ refers to the 'algebraic' nature. By definition, all elements of \mathbf{V} are *finite* linear combinations of elementary tensors $\mathbf{v} = \otimes_{j=1}^{d} v_{j} (v_{j} \in V_{j})$.

For vector spaces V_j and W_j over \mathbb{R} , let linear mappings $A_j : V_j \to W_j$ $(1 \le j \le d)$ be given. Then the definition of the elementary tensor

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j : \mathbf{V}_D = {}_a \bigotimes_{j=1}^{d} V_j \longrightarrow \mathbf{W}_D = {}_a \bigotimes_{j=1}^{d} W_j$$

is given by

$$\mathbf{A}\left(\bigotimes_{j=1}^{d} v_{j}\right) := \bigotimes_{j=1}^{d} \left(A_{j} v_{j}\right).$$

$$(1)$$

Note that (1) uniquely defines the linear mapping $\mathbf{A} : \mathbf{V}_D \to \mathbf{W}_D$. We recall that L(V, W) is the space of linear maps from V into W, while $V' = L(V, \mathbb{R})$ is the algebraic dual of V. For normed spaces, $\mathcal{L}(V, W)$ denotes the continuous linear maps, while $V^* = \mathcal{L}(V, \mathbb{R})$ is the topological dual of V.

2.2 Minimal subspaces in tensor representations

For a given $\alpha \in 2^D \setminus \{\emptyset, D\}$, we let $\mathbf{V}_{\alpha} := {}_a \bigotimes_{j \in \alpha} V_j$, with the convention $\mathbf{V}_{\{j\}} = V_j$ for all $j \in D$. The algebraic tensor space \mathbf{V}_D is identified with $\mathbf{V}_{\alpha} \otimes_a \mathbf{V}_{\alpha^c}$, where $\alpha^c = D \setminus \alpha$. For a tensor $\mathbf{v} \in \mathbf{V}_D = \mathbf{V}_{\alpha} \otimes_a \mathbf{V}_{\alpha^c}$, the minimal subspace $U_{\alpha}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha}$ of \mathbf{v} is defined by the properties that $\mathbf{v} \in U_{\alpha}^{\min}(\mathbf{v}) \otimes_a \mathbf{V}_{\alpha}$ and $\mathbf{v} \in \mathbf{U}_{\alpha} \otimes_a \mathbf{V}_{\alpha}$ implies $U_{\alpha}^{\min}(\mathbf{v}) \subset \mathbf{U}_{\alpha}$. Here we use the notation $U_{\{j\}}^{\min}(\mathbf{v}) = U_j^{\min}(\mathbf{v})$, and we adopt the convention $U_D^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{v}\}$. We recall some useful results on minimal subspaces (see Section 2.2 in [3]).

Proposition 1 Let $\mathbf{v} \in \mathbf{V}_D$. For any $\alpha \in 2^D \setminus \{\emptyset, D\}$, there exists a unique minimal subspace $U_{\alpha}^{\min}(\mathbf{v})$, where $\dim U_{\alpha}^{\min}(\mathbf{v}) < \infty$. Furthermore, it holds $\dim U_{\alpha}^{\min}(\mathbf{v}) = \dim U_{\alpha}^{\min}(\mathbf{v})$.

The relation between minimal subspaces is as follows (see Corollary 2.9 of [3]).

Proposition 2 Let $\mathbf{v} \in \mathbf{V}_D$. For any $\alpha \in 2^D$ with $\#\alpha \ge 2$ and a non-trivial partition \mathcal{P}_{α} of α , it holds

$$U^{\min}_{lpha}(\mathbf{v}) \subset {}_a \bigotimes_{eta \in \mathcal{P}_{lpha}} U^{\min}_{eta}(\mathbf{v})$$

Let \mathcal{P}_D be a given non-trivial partition of D. The algebraic tensor space $\mathbf{V}_D = {}_a \bigotimes_{j=1}^d V_j$ is identified with ${}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_{\alpha}$. By definition of the minimal subspaces $U_{\alpha}^{\min}(\mathbf{v}), \ \alpha \in \mathcal{P}_D$, we have

$$\mathbf{v} \in {}_a \bigotimes_{\alpha \in \mathcal{P}_D} U_\alpha^{\min}(\mathbf{v}) \; .$$

For a given $\alpha \in \mathcal{P}_D$ with $\#\alpha \geq 2$ and a non-trivial partition \mathcal{P}_α of α , we also have

$$\mathbf{v} \in \left(a \bigotimes_{\beta \in \mathcal{P}_{\alpha}} U_{\beta}^{\min}(\mathbf{v}) \right) \otimes_{a} \left(a \bigotimes_{\delta \in \mathcal{P}_{D} \setminus \{\alpha\}} U_{\delta}^{\min}(\mathbf{v}) \right).$$

The following result gives a characterisation of minimal subspaces.

Proposition 3 Let $\mathbf{v} \in \mathbf{V}_D$ and let α be a subset of D with $\#\alpha \geq 2$ and \mathcal{P}_{α} be a non-trivial partition of α . Assume that \mathbf{V}_{α} and \mathbf{V}_{β} , for $\beta \in \mathcal{P}_{\alpha}$, are normed spaces. Then for each $\beta \in \mathcal{P}_{\alpha}$, it holds

$$U_{\beta}^{\min}(\mathbf{v}) = \operatorname{span}\left\{ \left(id_{\beta} \otimes \boldsymbol{\varphi}^{(\alpha \setminus \beta)} \right) (\mathbf{v}_{\alpha}) : \mathbf{v}_{\alpha} \in U_{\alpha}^{\min}(\mathbf{v}), \ \boldsymbol{\varphi}^{(\alpha \setminus \beta)} \in {}_{a} \bigotimes_{\gamma \in \mathcal{P}_{\alpha} \setminus \{\beta\}} \mathbf{V}_{\beta}^{*} \right\}$$

Proof First observe that $\mathbf{V}_D = \mathbf{V}_{\alpha} \otimes_a \mathbf{V}_{\alpha^c} = \left(a \bigotimes_{\beta \in \mathcal{P}_{\alpha}} \mathbf{V}_{\beta} \right) \otimes_a \mathbf{V}_{\alpha^c}$. From [3, Theorem 2.17], we have $U_{\alpha}^{\min}(\mathbf{v}) = \left\{ (id_{\alpha} \otimes \boldsymbol{\varphi}^{(\alpha^c)})(\mathbf{v}) : \boldsymbol{\varphi}^{(\alpha^c)} \in \mathbf{V}_{\alpha^c}^* \right\}$. Since $\mathbf{v} \in \mathbf{V}_{\alpha} \otimes U_{\alpha^c}^{\min}(\mathbf{v})$, we can replace $\mathbf{V}_{\alpha^c}^*$ by the larger space $U_{\alpha^c}^{\min}(\mathbf{v})^*$, and obtain

$$U_{\alpha}^{\min}(\mathbf{v}) = \left\{ (id_{\alpha} \otimes \boldsymbol{\varphi}^{(\alpha^c)})(\mathbf{v}) : \boldsymbol{\varphi}^{(\alpha^c)} \in U_{\alpha^c}^{\min}(\mathbf{v})^* \right\}.$$

In a similar way and again from [3, Theorem 2.17], we also prove that for any $\beta \in \mathcal{P}_{\alpha}$, it holds

$$U_{\beta}^{\min}(\mathbf{v}) = \left\{ (id_{\beta} \otimes \boldsymbol{\varphi}^{(\beta^c)})(\mathbf{v}) : \boldsymbol{\varphi}^{(\beta^c)} \in \left(a \bigotimes_{\gamma \in \mathcal{P}_{\alpha} \setminus \{\beta\}} U_{\gamma}^{\min}(\mathbf{v})^* \right) \otimes_a U_{\alpha^c}^{\min}(\mathbf{v})^* \right\}$$

Take $\mathbf{v}_{\alpha} \in U_{\alpha}^{\min}(\mathbf{v})$. Then there exists $\boldsymbol{\varphi}^{(\alpha^c)} \in U_{\alpha^c}^{\min}(\mathbf{v})^*$ such that $\mathbf{v}_{\alpha} = (id_{\alpha} \otimes \boldsymbol{\varphi}^{(\gamma)})(\mathbf{v})$. Now, for $\boldsymbol{\varphi}^{(\alpha \setminus \beta)} \in {}_{a} \bigotimes_{\gamma \in \mathcal{P}_{\alpha} \setminus \{\beta\}} U_{\gamma}^{\min}(\mathbf{v})^*$, we have

$$\left(id_{\beta}\otimes\boldsymbol{\varphi}^{(\alpha\setminus\beta)}
ight)(\mathbf{v}_{lpha})=\left(id_{\beta}\otimes\boldsymbol{\varphi}^{(\alpha\setminus\beta)}\otimes\boldsymbol{\varphi}^{(lpha^{c})}
ight)(\mathbf{v}),$$

and hence $(id_{\beta} \otimes \varphi^{(\alpha \setminus \beta)})(\mathbf{v}_{\alpha}) \in U_{\beta}^{\min}(\mathbf{v})$. This proves a first inclusion. Now for $\beta \in \mathcal{P}_{\alpha}$, take $\mathbf{v}_{\beta} \in U_{\beta}^{\min}(\mathbf{v})$, then there exists

$$\boldsymbol{\varphi}^{(\beta^c)} \in \left(a \bigotimes_{\gamma \in \mathcal{P}_{\alpha} \setminus \{\beta\}} U_{\gamma}^{\min}(\mathbf{v})^* \right) \otimes_a U_{\alpha^c}^{\min}(\mathbf{v})^*$$

such that $\mathbf{v}_{\beta} = (id_{\beta} \otimes \boldsymbol{\varphi}^{(\beta^c)})(\mathbf{v})$. Then $\boldsymbol{\varphi}^{(\beta^c)} = \sum_{l=1}^r \boldsymbol{\psi}_l^{(\alpha \setminus \beta)} \otimes \boldsymbol{\phi}_l^{(\alpha^c)}$, where $\boldsymbol{\phi}_l^{(\alpha^c)} \in U_{\alpha^c}^{\min}(\mathbf{v})^*$ and $\boldsymbol{\psi}_l^{(\alpha \setminus \beta)} \in {}_a \bigotimes_{\gamma \in \mathcal{P}_{\alpha \setminus \{\beta\}}} U_{\gamma}^{\min}(\mathbf{v})^*$, for $1 \leq l \leq r$. Thus,

$$\mathbf{v}_{\beta} = \left(id_{\beta} \otimes \boldsymbol{\varphi}^{(\beta^{c})}\right)(\mathbf{v}) = \sum_{i=1}^{r} \left(id_{\beta} \otimes \boldsymbol{\psi}_{i}^{(\alpha \setminus \beta)} \otimes \boldsymbol{\phi}_{i}^{(\alpha^{c})}\right)(\mathbf{v})$$
$$= \sum_{i=1}^{r} \left(id_{\beta} \otimes \boldsymbol{\psi}_{i}^{(\alpha \setminus \beta)}\right) \left(\left(id_{\alpha} \otimes \boldsymbol{\phi}_{i}^{(\alpha^{c})}\right)(\mathbf{v})\right).$$

Observing that $(id_{\alpha} \otimes \phi_l^{(\alpha^c)})(\mathbf{v}) \in U_{\alpha}^{\min}(\mathbf{v})$, we obtain the other inclusion. \Box

2.3 Algebraic tensor spaces in the tree-based format

Definition 1 A tree T_D is called a dimension partition tree of D if

- (a) all vertices $\alpha \in T_D$ are non-empty subsets of D,
- (b) D is the root of T_D ,
- (c) every vertex $\alpha \in T_D$ with $\#\alpha \ge 2$ has at least two sons and the set of sons of α , denoted $S(\alpha)$, is a non-trivial partition of α ,
- (d) every vertex $\alpha \in T_D$ with $\#\alpha = 1$ has no son and is called a *leaf*.

The set of leaves is denoted by $\mathcal{L}(T_D)$. A straightforward consequence of Definition 1 is that the set of leaves $\mathcal{L}(T_D)$ coincides with the singletons of D, i.e., $\mathcal{L}(T_D) = \{\{j\} : j \in D\}$ and hence it is the trivial partition of D. We remark that for a tree T_D such that $S(D) \neq \mathcal{L}(T_D)$, S(D) is a non-trivial partition of D.

We denote by $\operatorname{level}(\alpha)$, $\alpha \in T_D$, the levels of the vertices in T_D , which are such that $\operatorname{level}(D) = 0$ and for any pair $\alpha, \beta \in T_D$ such that $\beta \in S(\alpha)$, $\operatorname{level}(\beta) = \operatorname{level}(\alpha) + 1$. The $depth^1$ of the tree T_D is defined as $\operatorname{depth}(T_D) = \max_{\alpha \in T_D} \operatorname{level}(\alpha)$.

Definition 2 For a tensor space \mathbf{V}_D and a dimension partition tree T_D , the pair (\mathbf{V}_D, T_D) is called a representation of the tensor space \mathbf{V}_D in *tree-based format*, and is associated with the collection of spaces $\{\mathbf{V}_{\alpha}\}_{\alpha \in T_D \setminus D}$.

Example 1 (Tucker format) In Figure 1, $D = \{1, 2, 3, 4, 5, 6\}$ and

$$T_D = \{D, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}, \{6\}\}$$

Here depth $(T_D) = 1$. This corresponds to the Tucker format.

 $^{^{1}}$ By using the notion of edge, that is, the connection between one vertex to another, then our definition of depth coincides with the classical definition of height, i.e. the longest downward path between the root and a leaf.



Fig. 1 Tuker format: dimension partition tree depth $(T_D) = 1$ with $S(D) = \mathcal{L}(T_D)$.

Example 2 In Figure 2, $D = \{1, 2, 3, 4, 5, 6\}$ and

 $T_D = \{D, \{1, 2, 3\}, \{4, 5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$

Here depth $(T_D) = 2$.



Fig. 2 A dimension partition tree with depth $(T_D) = 2$.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For each $\mathbf{v} \in \mathbf{V}_D$, we have that $(\dim U^{\min}_{\alpha}(\mathbf{v}))_{\alpha \in 2^D \setminus \{\emptyset\}}$ is in $\mathbb{N}_0^{2^{\# D} - 1}$.

Definition 3 For a given partition dimension tree T_D over D, and for each $\mathbf{v} \in \mathbf{V}_D$, we define its *tree-based rank* by the tuple $\operatorname{rank}_{T_D}(\mathbf{v}) := (\dim U^{\min}_{\alpha}(\mathbf{v}))_{\alpha \in T_D} \in \mathbb{N}_0^{\#T_D}$.

Definition 4 We will say that $\mathfrak{r} := (r_{\alpha})_{\alpha \in T_D} \in \mathbb{N}^{\#T_D}$ is an *admissible tuple* for T_D , if there exists $\mathbf{v} \in \mathbf{V}_D$ such that $\dim U_{\alpha}^{\min}(\mathbf{v}) = r_{\alpha}$ for all $\alpha \in T_D$. We will denote the set of admissible ranks for the representation (\mathbf{V}_D, T_D) of the tensor space \mathbf{V}_D by

$$\mathcal{AD}(\mathbf{V}_D, T_D) := \{ (\dim U^{\min}_{\alpha}(\mathbf{v}))_{\alpha \in T_D} : \mathbf{v} \in \mathbf{V}_D \}$$

2.4 The set of tensors in tree-based format with fixed or bounded tree-based rank

Definition 5 Let T_D be a given dimension partition tree and fix some tuple $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$. Then the set of tensors of fixed tree-based rank \mathfrak{r} is defined by

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) := \left\{ \mathbf{v} \in \mathbf{V}_D : \dim U^{\min}_{\alpha}(\mathbf{v}) = r_{\alpha} \text{ for all } \alpha \in T_D \right\}$$
(2)

and the set of tensors of tree-based rank bounded by \mathfrak{r} is defined by

$$\mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D, T_D) := \left\{ \mathbf{v} \in \mathbf{V}_D : \dim U_{\alpha}^{\min}(\mathbf{v}) \leq r_{\alpha} \text{ for all } \alpha \in T_D \right\}.$$
(3)

For $\mathfrak{r}, \mathfrak{s} \in \mathbb{N}_0^{\#T_D}$ we write $\mathfrak{s} \leq \mathfrak{r}$ if and only if $s_{\alpha} \leq r_{\alpha}$ for all $\alpha \in T_D$. Then fo a fixed $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$, we have

$$\mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D, T_D) := \bigcup_{\substack{\mathfrak{s} \leq \mathfrak{r} \\ \mathfrak{s} \in \mathcal{AD}(\mathbf{V}_D, T_D)}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D, T_D).$$
(4)

We point out that in [2] is introduced a representation of \mathbf{V}_D in Tucker format. Letting T_D^{Tucker} be the Tucker dimension partition tree (see example 1) and given $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D^{\text{Tucker}})$, we define the set of tensors with fixed Tucker rank \mathfrak{r} by

$$\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D) := \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D^{\mathrm{Tucker}}) = \left\{ \mathbf{v} \in \mathbf{V}_D : \dim U_k^{\mathrm{min}}(\mathbf{v}) = r_k, \, k \in \mathcal{L}(T_D^{\mathrm{Tucker}}) \right\}.$$

Then

$$\mathbf{V}_D = igcup_{\mathfrak{r}\in\mathcal{AD}(\mathbf{V}_D,T_D^{ ext{Tucker}})}\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D).$$

 $2.5~\mathrm{The}$ representation of tensors in tree based form at with fixed tree based rank

Before stating the next result we recall the definition of the 'matricisation' (or 'unfolding') of a tensor in a finite-dimensional setting.

Definition 6 Let α be a finite set of indices, \mathcal{P}_{α} be a non-trivial partition of α , and $r = (r_{\mu})_{\mu \in \mathcal{P}_{\alpha}} \in \mathbb{N}^{\#\mathcal{P}_{\alpha}}$. For $\beta \in \mathcal{P}_{\alpha}$, we define a map \mathcal{M}_{β}

$$\mathcal{M}_{\beta}: \begin{array}{ll} \mathbb{R}^{\times_{\mu\in\mathcal{P}_{\alpha}}r_{\mu}} & \to & \mathbb{R}^{r_{\beta}\times\left(\prod_{\mu\in\mathcal{P}_{\alpha}\setminus\{\beta\}}r_{\mu}\right)}, \\ C_{(i_{\mu})_{\mu\in\mathcal{P}_{\alpha}}} & \mapsto & C_{i_{\beta},(i_{\mu})_{\mu\in\mathcal{P}_{\alpha}\setminus\{\beta\}}} \end{array}$$

which is an isomorphism. Given $C \in \mathbb{R}^{\times_{\mu \in \mathcal{P}_{\alpha}} r_{\mu}}$ we have that $C \in \mathfrak{M}_r \left(\mathbb{R}^{\times_{\mu \in \mathcal{P}_{\alpha}} r_{\mu}} \right)$ if and only if rank $\mathcal{M}_{\beta}(C) = r_{\beta}$ for each $\beta \in \mathcal{P}_{\alpha}$, or equivalently $\mathcal{M}_{\beta}(C)\mathcal{M}_{\beta}(C)^T \in \mathrm{GL}(\mathbb{R}^{r_{\beta}})$ for $\beta \in \mathcal{P}_{\alpha}$.

The next result gives us a characterisation of the tensors in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$.

Theorem 1 Let T_D be a dimension partition tree over D with depth $(T_D) = \mathfrak{d}$. Given $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$ then the following statements are equivalent.

(a)
$$\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D).$$

(b) Given $\{u_{i_k}^{(k)} : 1 \le i_k \le r_k\}$ a fixed basis of $U_k^{\min}(\mathbf{v})$ for $k \in \mathcal{L}(T_D),$

$$\mathbf{v} = \sum_{\substack{1 \le i_{\alpha} \le r_{\alpha} \\ \alpha \in S(D)}} C_{(i_{\alpha})_{\alpha \in S(D)}}^{(D)} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_{\alpha}}^{(\alpha)},$$
(5)

for a unique $C^{(D)} \in \mathfrak{M}_r(\mathbb{R}^{\times_{\beta \in S(D)} r_\beta})$ and where for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$, there exists a unique $C^{(\mu)} \in \mathbb{R}^{r_\mu \times \times_{\beta \in S(\mu)} r_\beta}$ such that rank $\mathcal{M}_\mu(C^{(\mu)}) = \dim U^{\min}_\mu(\mathbf{v}) = r_\mu$, and the set $\{\mathbf{u}^{(\mu)}_{i_\mu} : 1 \leq i_\mu \leq r_\mu\}$, with

$$\mathbf{u}_{i_{\mu}}^{(\mu)} = \sum_{\substack{1 \le i_{\beta} \le r_{\beta} \\ \beta \in S(\mu)}} C_{i_{\mu},(i_{\beta})_{\beta \in S(\mu)}}^{(\mu)} \bigotimes_{\beta \in S(\mu)} \mathbf{u}_{i_{\beta}}^{(\beta)}$$
(6)

for $1 \leq i_{\mu} \leq r_{\mu}$, is a basis of $U_{\mu}^{\min}(\mathbf{v})$.

Proof (b) clearly implies (a). Now consider $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$. Since $\mathbf{v} \in \bigotimes_{\alpha \in S(D)} U_{\alpha}^{\min}(\mathbf{v})$, there exists a unique $C^{(D)} \in \mathbb{R}^{\times_{\beta \in S(D)} r_{\beta}}$ such that

$$\mathbf{v} = \sum_{\substack{1 \le i_{\alpha} \le r_{\alpha} \\ \alpha \in S(D)}} C_{(i_{\alpha})_{\alpha \in S(D)}}^{(D)} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_{\alpha}}^{(\alpha)},$$

where $\{\mathbf{u}_{i_{\alpha}}^{(\alpha)}: 1 \leq i_{\alpha} \leq r_{\alpha}\}$ is a fixed basis of $U_{\alpha}^{\min}(\mathbf{v})$ for $\alpha \in S(D)$. Since rank $\mathcal{M}_{\alpha}(C^{(D)}) = \dim U_{\alpha}^{\min}(\mathbf{v}) = r_{\alpha}$ for each $\alpha \in S(D)$, we have that $C^{(D)} \in \mathfrak{M}_{r}(\mathbb{R}^{\times_{\beta \in S(D)} r_{\beta}})$. Now, for each $\mu \in T_{D} \setminus \{D\}$ such that $S(\mu) \neq \emptyset$, thanks to Proposition 2, we have

$$U^{\min}_{\mu}(\mathbf{v}) \subset {}_{a} \bigotimes_{eta \in S(\mu)} U^{\min}_{eta}(\mathbf{v}) \; .$$

Consider $\{\mathbf{u}_{i_{\mu}}^{(\mu)} : 1 \leq i_{\mu} \leq r_{\mu}\}$ a basis of $U_{\mu}^{\min}(\mathbf{v})$ and $\{\mathbf{u}_{i_{\beta}}^{(\beta)} : 1 \leq i_{\beta} \leq r_{\beta}\}$ a basis of $U_{\beta}^{\min}(\mathbf{v})$ for $\beta \in S(\mu)$ and $1 \leq i_{\mu} \leq r_{\mu}$. Then, there exists a unique $C^{(\mu)} \in \mathbb{R}^{r_{\mu} \times (\chi_{\beta \in S(\alpha)}, r_{\beta})}$ such that

$$\mathbf{u}_{i_{\mu}}^{(\mu)} = \sum_{\substack{1 \le i_{\beta} \le r_{\beta} \\ \beta \in S(\mu)}} C_{i_{\mu},(i_{\beta})_{\beta \in S(\mu)}}^{(\mu)} \bigotimes_{\beta \in S(\mu)} \mathbf{u}_{i_{\beta}}^{(\beta)},$$

for $1 \le i_{\mu} \le r_{\mu}$. Since $\{\mathbf{u}_{i_{\mu}}^{(\mu)} : 1 \le i_{\mu} \le r_{\mu}\}$ is a basis, then

$$\operatorname{rank} \mathcal{M}_{\mu}(C^{(\mu)}) = \dim U_{\mu}^{\min}(\mathbf{v}) = r_{\mu},\tag{7}$$

holds for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$. Then (c) holds.

3 Topological tensor spaces in the tree-based format

First, we recall the definition of tensor Banach spaces.

Definition 7 We say that $\mathbf{V}_{\|\cdot\|}$ is a *Banach tensor space* if there exists an algebraic tensor space \mathbf{V} and a norm $\|\cdot\|$ on \mathbf{V} such that $\mathbf{V}_{\|\cdot\|}$ is the completion of \mathbf{V} with respect to the norm $\|\cdot\|$, i.e.

$$\mathbf{V}_{\|\cdot\|} := \|\cdot\| \bigotimes_{j=1}^{d} V_j = \overline{a \bigotimes_{j=1}^{d} V_j}^{\|\cdot\|}.$$

If $\mathbf{V}_{\|\cdot\|}$ is a Hilbert space, we say that $\mathbf{V}_{\|\cdot\|}$ is a *Hilbert tensor space*.

Next, we give some examples of Banach and Hilbert tensor spaces.

Example 3 For $I_j \subset \mathbb{R}$ $(1 \leq j \leq d)$ and $1 \leq p < \infty$, the Sobolev space $H^{N,p}(I_j)$ consists of all univariate functions f from $L^p(I_j)$ with bounded norm²

$$||f||_{N,p;I_j} := \left(\sum_{n=0}^N \int_{I_j} |\partial^n f|^p \, \mathrm{d}x\right)^{1/p},$$

whereas the space $H^{N,p}(\mathbf{I})$ of *d*-variate functions on $\mathbf{I} = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ is endowed with the norm

$$\|f\|_{N,p} := \left(\sum_{0 \le |\mathbf{n}| \le N} \int_{\mathbf{I}} |\partial^{\mathbf{n}} f|^{p} \,\mathrm{d}\mathbf{x}\right)^{1/p}$$

with $\mathbf{n} \in \mathbb{N}_0^d$ being a multi-index of length $|\mathbf{n}| := \sum_{j=1}^d n_j$. For p > 1 it is well known that $H^{N,p}(I_j)$ and $H^{N,p}(\mathbf{I})$ are reflexive and separable Banach spaces. Moreover, for p = 2, the Sobolev spaces $H^N(I_j) := H^{N,2}(I_j)$ and $H^N(\mathbf{I}) := H^{N,2}(\mathbf{I})$ are Hilbert spaces. As a first example,

$$H^{N,p}(\mathbf{I}) = \lim_{\|\cdot\|_{N,p}} \bigotimes_{j=1}^{d} H^{N,p}(I_j)$$

is a Banach tensor space. Examples of Hilbert tensor spaces are

$$L^{2}(\mathbf{I}) = \lim_{\|\cdot\|_{0,2}} \bigotimes_{j=1}^{d} L^{2}(I_{j}) \quad \text{and} \quad H^{N}(\mathbf{I}) = \lim_{\|\cdot\|_{N,2}} \bigotimes_{j=1}^{d} H^{N}(I_{j}) \text{ for } N \in \mathbb{N}.$$

In the definition of a tensor Banach space $\|\cdot\| \bigotimes_{j \in D} V_j$ we have not fixed whether the V_j , for $j \in D$, are complete or not. This leads us to introduce the following definition.

Definition 8 Let D be a finite index set and T_D be a dimension partition tree over D. Let $(V_j, \|\cdot\|_j)$ be a normed space such that $V_{j\|\cdot\|_j}$ is a Banach space obtained by the completion of V_j , for $j \in D$, and consider a representation $\{\mathbf{V}_{\alpha}\}_{\alpha \in T_D \setminus \{D\}}$ of the tensor space $\mathbf{V}_D = a \bigotimes_{j \in D} V_j$ where for each $\alpha \in T_D \setminus$

² It suffices to have in (8) the terms n = 0 and n = N. The derivatives are to be understood as weak derivatives.

 $\mathcal{L}(T_D)$ we have a tensor space $\mathbf{V}_{\alpha} = {}_{a} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta}$. If for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$ there exists a norm $\|\cdot\|_{\alpha}$ defined on \mathbf{V}_{α} such that $\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta}$ is a tensor Banach space, we say that $\{\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = \|\cdot\|_D \bigotimes_{j \in D} V_j$ in the topological tree-based format.

For $\alpha \in T_D \setminus \mathcal{L}(T_D)$,

$$\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}} = \lim_{\|\cdot\|_{\alpha}} \bigotimes_{j \in \alpha} V_j = \lim_{\|\cdot\|_{\alpha}} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta} .$$

Example 4 Figure 3 gives an example of a representation in the topological tree-based format for an anisotropic Sobolev space.



Fig. 3 A representation in the topological tree-based format for the tensor Banach space $\overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}}$. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

Remark 1 Observe that the example in Figure 4 is not included in the definition of the topological tree-based format. Moreover, for a tensor $\mathbf{v} \in L^p(I_1) \otimes_a (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3))$, we have $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3)$. However, in the topological tree-based representation of Figure 3, for a given $\mathbf{v} \in L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$ we have $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$, and hence $U_{23}^{\min}(\mathbf{v}) \subset U_2^{\min}(\mathbf{v}) \otimes_a U_3^{\min}(\mathbf{v})$.

The difference between the tensor spaces involved in Figure 3 and Figure 4 is given by the fact that since

$$H^{N,p}(I_2) \otimes_a H^{N,p}(I_3) \subset \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}$$

then

$$\overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}} \subset \overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}}^{\|\cdot\|_{23}}$$

A desirable property for the tensor product is that if $\|\cdot\|_{\alpha}$ for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$ is a norm on the tensor space $_a \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta \|\cdot\|_{\beta}}$, then

$$\|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta\|\cdot\|_{\beta}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta} = \|\cdot\|_{\alpha} \bigotimes_{j \in \alpha} V_{j}$$
(9)



Fig. 4 A representation for the tensor Banach space $\overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}}$, using a tree. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

must be true. To precise these ideas, we introduce the following definitions and results.

Let $\|\cdot\|_j$, $1 \leq j \leq d$, be the norms of the vector spaces V_j appearing in $\mathbf{V}_D = {}_a \bigotimes_{j=1}^d V_j$. By $\|\cdot\|_D$ we denote the norm on the tensor space \mathbf{V}_D . Note that $\|\cdot\|_D$ is not determined by $\|\cdot\|_j$, for $j \in D$, but there are relations which are 'reasonable'. Any norm $\|\cdot\|$ on ${}_a \bigotimes_{j=1}^d V_j$ satisfying

$$\left\|\bigotimes_{j=1}^{d} v_{j}\right\| = \prod_{j=1}^{d} \|v_{j}\|_{j} \quad \text{for all } v_{j} \in V_{j} \ (1 \le j \le d) \quad (10)$$

is called a *crossnorm*. As usual, the dual norm of $\|\cdot\|$ is denoted by $\|\cdot\|^*$. If $\|\cdot\|$ is a crossnorm and also $\|\cdot\|^*$ is a crossnorm on $_a \bigotimes_{j=1}^d V_j^*$, i.e.,

$$\left\|\bigotimes_{j=1}^{d}\varphi^{(j)}\right\|^{*} = \prod_{j=1}^{d} \|\varphi^{(j)}\|_{j}^{*} \quad \text{for all } \varphi^{(j)} \in V_{j}^{*} \ (1 \le j \le d), \quad (11)$$

then $\|\cdot\|$ is called a *reasonable crossnorm*.

Remark 2 Eq. (10) implies the inequality $\|\bigotimes_{j=1}^{d} v_{j}\| \lesssim \prod_{j=1}^{d} \|v_{j}\|_{j}$ which is equivalent to the continuity of the multilinear tensor product mapping³ between normed spaces:

$$\bigotimes : \bigotimes_{j=1}^{d} \left(V_{j}, \|\cdot\|_{j} \right) \longrightarrow \left(a \bigotimes_{j=1}^{d} V_{j}, \|\cdot\| \right),$$
(12)

defined by $\bigotimes ((v_1, \ldots, v_d)) = \bigotimes_{j=1}^d v_j$, the product space being equipped with the product topology induced by the maximum norm $||(v_1, \ldots, v_d)|| = \max_{1 \le j \le d} ||v_j||_j$.

³ Recall that a multilinear map T from $X_{j=1}^{d}(V_{j}, \|\cdot\|_{j})$ equipped with the product topology to a normed space $(W, \|\cdot\|)$ is continuous if and only if $\|T\| < \infty$, with

 $^{\|}T\| := \sup_{\substack{(v_1, \dots, v_d) \\ \|(v_1, \dots, v_d)\| \le 1}} \|T(v_1, \dots, v_d)\| = \sup_{\substack{(v_1, \dots, v_d) \\ \|v_1\|_1 \le 1, \dots, \|v_d\|_d \le 1}} \|T(v_1, \dots, v_d)\| = \sup_{(v_1, \dots, v_d)} \frac{\|T(v_1, \dots, v_d)\|}{\|v_1\|_1 \dots \|v_d\|_d}$

The following result is a consequence of Lemma 4.34 of [9].

Lemma 1 Let $(V_j, \|\cdot\|_j)$ be normed spaces for $1 \le j \le d$. Assume that $\|\cdot\|$ is a norm on the tensor space ${}_a \bigotimes_{j=1}^d V_{j_{\|\cdot\|_j}}$ such that the tensor product map

$$\bigotimes: \bigotimes_{j=1}^{d} \left(V_{j_{\|\cdot\|_{j}}}, \|\cdot\|_{j} \right) \longrightarrow \left(a \bigotimes_{j=1}^{d} V_{j_{\|\cdot\|_{j}}}, \|\cdot\| \right)$$
(13)

is continuous. Then (12) is also continuous and

$$\|\cdot\|\bigotimes_{j=1}^{d} V_{j\|\cdot\|_{j}} = \|\cdot\|\bigotimes_{j=1}^{d} V_{j}$$

holds.

Definition 9 Assume that for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$ there exists a norm $\|\cdot\|_{\alpha}$ defined on ${}_a \bigotimes_{\beta \in S(\alpha)} V_{\beta_{\|\cdot\|_{\beta}}}$. We will say that the tensor product map \bigotimes is T_D -continuous if the map

$$\bigotimes: \bigotimes_{\beta \in S(\alpha)} (V_{\beta_{\|\cdot\|_{\beta}}}, \|\cdot\|_{\beta}) \to \left(a \bigotimes_{\beta \in S(\alpha)} V_{\beta_{\|\cdot\|_{\beta}}}, \|\cdot\|_{\alpha} \right)$$

is continuous for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

The next result gives the conditions to have (9).

Theorem 2 Assume that we have a representation $\{\mathbf{V}_{\alpha_{\parallel}\cdot\parallel_{\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ in the topological tree-based format of the tensor Banach space $\mathbf{V}_{D_{\parallel}\cdot\parallel_{D}} = {}_{\parallel\cdot\parallel_{D}} \bigotimes_{\alpha\in S(D)} \mathbf{V}_{\alpha}$, such that for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$, the norm ${}_{\parallel}\cdot\parallel_{\alpha}$ is also defined on ${}_{a} \bigotimes_{\beta\in S(\alpha)} V_{\beta_{\parallel}\cdot\parallel_{\beta}}$ and the tensor product map \bigotimes is T_D -continuous. Then

$$\|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta\|\cdot\|_{\beta}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta} = \|\cdot\|_{\alpha} \bigotimes_{j \in \alpha} V_{j} ,$$

holds for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

Proof From Lemma 1, if the tensor product map

$$\bigotimes : \bigotimes_{\beta \in S(\alpha)} (\mathbf{V}_{\beta_{\|\cdot\|_{\beta}}}, \|\cdot\|_{\beta}) \longrightarrow (\underset{\beta \in S(\alpha)}{\bigotimes} \mathbf{V}_{\beta_{\|\cdot\|_{\beta}}}, \|\cdot\|_{\alpha})$$

is continuous, then

$$\|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_{\beta}}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} V_{\beta} ,$$

holds. Since $\mathbf{V}_{\alpha} = {}_{a} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta} = {}_{a} \bigotimes_{j \in \alpha} V_{j}$, the theorem follows. \Box

Example 5 Assume that the tensor product maps

$$\bigotimes : (L^{p}(I_{1}), \|\cdot\|_{0,p;I_{1}}) \times (H^{N,p}(I_{2}) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_{3}), \|\cdot\|_{23}) \to (L^{p}(I_{1}) \otimes_{a} (H^{N,p}(I_{2}) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_{3})), \|\cdot\|_{123})$$

$$\bigotimes : (H^{N,p}(I_2), \|\cdot\|_{N,p;I_2}) \times (H^{N,p}(I_3), \|\cdot\|_{N,p;I_3}) \to (H^{N,p}(I_2) \otimes_a H^{N,p}(I_3), \|\cdot\|_{23})$$

are continuous. Then the trees of Figure 3 and Figure 4 are the same.

4 On the best approximation in $\mathcal{FT}_{\leq \mathfrak{r}}(V_D)$

Now we discuss about the best approximation problem in $\mathcal{FT}_{\leq \mathbf{r}}(\mathbf{V}_D)$. For this, we need a stronger condition than the T_D -continuity of the tensor product. Grothendieck [7] named the norm $\|\cdot\|_{\vee}$ introduced below the *injective norm*.

Definition 10 Let V_i be a Banach space with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. Then for $\mathbf{v} \in \mathbf{V} = {}_a \bigotimes_{j=1}^d V_j$ define the norm $\|\cdot\|_{\vee(V_1,\ldots,V_d)}$, called the injective norm, by

$$\|\mathbf{v}\|_{\vee(V_1,\dots,V_d)} := \sup\left\{\frac{|(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_d)(\mathbf{v})|}{\prod_{j=1}^d \|\varphi_j\|_j^*} : 0 \neq \varphi_j \in V_j^*, 1 \le j \le d\right\}.$$
(14)

It is well known that the injective norm is a reasonable crossnorm (see Lemma 1.6 in [12] and (10)-(11)). Further properties are given by the next proposition (see Lemma 4.96 and 4.2.4 in [9]).

Proposition 4 Let V_i be a Banach space with norm $\|\cdot\|_i$ for $1 \leq i \leq d$, and $\|\cdot\|$ be a norm on $\mathbf{V} := {}_a \bigotimes_{j=1}^d V_j$. The following statements hold.

(a) For each $1 \leq j \leq d$ introduce the tensor Banach space

$$\mathbf{X}_j := \lim_{\|\cdot\|_{\vee(V_1,\ldots,V_{j-1},V_{j+1},\ldots,V_d)}} \bigotimes_{k \neq j} V_k \; .$$

Then

$$\|\cdot\|_{\vee(V_1,...,V_d)} = \|\cdot\|_{\vee(V_j,\mathbf{X}_j)}$$
(15)

holds for $1 \leq j \leq d$.

(b) The injective norm is the weakest reasonable crossnorm on \mathbf{V} , i.e., if $\|\cdot\|$ is a reasonable crossnorm on \mathbf{V} , then

$$\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\dots,V_d)}. \tag{16}$$

(c) For any norm $\|\cdot\|$ on **V** satisfying $\|\cdot\|_{\vee(V_1,\ldots,V_d)} \lesssim \|\cdot\|$, the map (12) is continuous, and hence Fréchet differentiable.

In Corollary 4.4 in [3] the following result, which is re-stated here using the notations of the present paper, is proved as a consequence of a similar result showed for tensors in Tucker format with bounded rank.

Theorem 3 Let $\mathbf{V}_D = {}_a \bigotimes_{j \in D} V_j$ and let $\{\mathbf{V}_{\alpha_j \parallel \cdot \parallel \alpha_j} : 2 \leq j \leq d\} \cup \{V_{j_{\parallel \cdot \parallel j}} : 1 \leq j \leq d\}$ for $d \geq 3$, be a representation of a reflexive Banach tensor space $\mathbf{V}_{D_{\parallel \cdot \parallel_D}} = {}_{\parallel \cdot \parallel_D} \bigotimes_{j \in D} V_j$, in topological tree-based format such that

(a) $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(V_{1_{\|\cdot\|_{j}}},...,V_{d_{\|\cdot\|_{d}}})},$ (b) $\mathbf{V}_{\alpha_d} = V_{d-1} \otimes_a V_d, \text{ and } \mathbf{V}_{\alpha_j} = V_{j-1} \otimes_a \mathbf{V}_{\alpha_{j+1}}, \text{ for } 2 \le j \le d-1, \text{ and}$ (c) $\|\cdot\|_{\alpha_j} := \|\cdot\|_{\vee(V_{j-1_{\|\cdot\|_{j-1}}},...,V_{d_{\|\cdot\|_{d}}})}$ for $2 \le j \le d$.

Then for each $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_{D}}}$ there exists $\mathbf{u}_{best} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_{D})$ such that

$$\|\mathbf{v} - \mathbf{u}_{best}\|_{D} = \min_{\mathbf{u} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_{D})} \|\mathbf{v} - \mathbf{u}\|_{D}.$$

It seems clear that tensor Banach spaces as we show in Example 4 are not included in this framework. So a natural question is to ask if for a representation in the topological tree-based format of a reflexive Banach space the statement of Theorem 3 is also true. To prove this, we will reformulate some of the results given in [3]. In the aforementioned paper, the milestone to prove the existence of a best approximation is the extension of the definition of minimal subspaces for tensors $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$. To do this the authors use a result similar to the following lemma (see Lemma 3.8 in [3]).

Lemma 2 Let $V_{j_{\|\cdot\|_{j}}}$ be a Banach space for $j \in D$, where D is a finite index set, and $\alpha_{1}, \ldots, \alpha_{m} \subset 2^{D} \setminus \{D, \emptyset\}$, be such that $\alpha_{i} \cap \alpha_{j} = \emptyset$ for all $i \neq j$ and $D = \bigcup_{i=1}^{m} \alpha_{i}$. Assume that if $\#\alpha_{i} \geq 2$ for some $1 \leq i \leq m$, then $\mathbf{V}_{\alpha_{i}}$ is a tensor Banach space. Consider the tensor space

$$\mathbf{V}_D := {}_a \bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}{}_{\|\cdot\|_{\alpha_i}}$$

endowed with the injective norm $\|\cdot\|_{\vee(\mathbf{V}_{\alpha_{1}}\|\cdot\|_{\alpha_{1}},...,\mathbf{V}_{\alpha_{m}}\|\cdot\|_{\alpha_{m}})}$. Fix $1 \leq k \leq m$, then given $\varphi_{[\alpha_{k}]} \in a \bigotimes_{i \neq k} \mathbf{V}_{\alpha_{i}}^{*}$ the map $id_{\alpha_{k}} \otimes \varphi_{[\alpha_{k}]}$ belongs to $\mathcal{L}\left(\mathbf{V}_{D}, \mathbf{V}_{\alpha_{k}}\|\cdot\|_{\alpha_{k}}\right)$. Moreover, $\overline{id_{\alpha_{k}} \otimes \varphi_{[\alpha_{k}]}} \in \mathcal{L}(\overline{\mathbf{V}_{D}}^{\|\cdot\|}, \mathbf{V}_{\alpha_{k}}\|\cdot\|_{\alpha_{k}})$ for any norm satisfying $\|\cdot\| \gtrsim \|\cdot\|_{\vee(\mathbf{V}_{\alpha_{1}}\|\cdot\|_{\alpha_{1}},...,\mathbf{V}_{\alpha_{m}}\|\cdot\|_{\alpha_{m}})$.

Let $\{\mathbf{V}_{\alpha_{\parallel},\parallel_{\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\parallel},\parallel_{D}} = \underset{\parallel,\parallel_{D}}{\parallel} \bigotimes_{j\in D} V_{j}$, in the topological tree-based format and assume that the tensor product map \bigotimes is T_{D} -continuous. From Theorem 2, we may assume that we have a tensor Banach space

$$\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}} = \lim_{\|\cdot\|_{\alpha}} \bigotimes_{\beta \in S(\alpha)} V_{\beta_{\|\cdot\|_{\beta}}}$$

for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$, and a Banach space $V_{j_{\|\cdot\|_j}}$ for $j \in \mathcal{L}(T_D)$. Let $\alpha \in T_D \setminus \mathcal{L}(T_D)$. To simplify the notation we write for $A, B \subset S(\alpha)$

$$\|\cdot\|_{\vee(A)} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_{\delta}}}:\delta\in A\})}$$

and

$$\|\cdot\|_{\vee(A,\vee(B))} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_{\delta}}}:\delta\in A\},\mathbf{X}_B\}}$$

where

$$\mathbf{X}_B := \lim_{\|\cdot\|_{\vee(B)}} \bigotimes_{eta \in B} \mathbf{V}_{eta_{\|\cdot\|_{eta}}}$$

From Proposition 4(a), we can write

$$\|\cdot\|_{\vee(S(\alpha))} = \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$$

for each $\beta \in S(\alpha)$. From now on, we assume that

$$\|\cdot\|_{\alpha} \gtrsim \|\cdot\|_{\vee(S(\alpha))} \text{ for each } \alpha \in T_D \setminus \mathcal{L}(T_D), \tag{17}$$

holds. Recall that Proposition 4(c) implies that the tensor product map \bigotimes is T_D -continuous. Since $\|\cdot\|_{\alpha} \gtrsim \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$ holds for each $\beta \in S(\alpha)$, the tensor product map

$$\bigotimes : (\mathbf{V}_{\beta_{\|\cdot\|_{\beta}}}, \|\cdot\|_{\beta}) \times \left(\underset{\delta \in S(\alpha) \setminus \{\beta\}}{\bigotimes} \mathbf{V}_{\delta_{\|\cdot\|_{\delta}}}, \|\cdot\|_{\vee(S(\alpha) \setminus \beta)} \right) \to (\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}, \|\cdot\|_{\alpha})$$

is also continuous for each $\beta \in S(\alpha)$. Moreover, by Theorem 2,

$$\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} V_{\beta_{\|\cdot\|_{\beta}}} = \|\cdot\|_{\alpha} \bigotimes_{\beta \in S(\alpha)} V_{\beta} = \|\cdot\|_{\alpha} \bigotimes_{j \in \alpha} V_{j} ,$$

holds for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$. Observe, that $\mathbf{V}^*_{\alpha_{\|\cdot\|_{\alpha}}} \subset \mathbf{V}^*_{\alpha}$ for all $\alpha \in S(D)$. Take $\mathbf{V}_D = {}_a \bigotimes_{j \in D} V_j$. Since $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(S(D))}$, from Lemma 2 and Proposition 3, we can extend for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$, the definition of minimal subspace for each $\alpha \in S(D)$ as

$$U_{\alpha}^{\min}(\mathbf{v}) := \left\{ \overline{(id_{\alpha} \otimes \varphi_{[\alpha]})}(\mathbf{v}) : \varphi_{[\alpha]} \in {}_{a} \bigotimes_{\beta \in S(D) \setminus \{\alpha\}} \mathbf{V}_{\beta}^{*} \right\}.$$

Observe that $\overline{(id_{\alpha} \otimes \varphi_{[\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_{D}}}, \mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}})$. Recall that if $\mathbf{v} \in \mathbf{V}_{D}$ and $\alpha \notin \mathcal{L}(T_{D})$, from Proposition 2, we have $U_{\alpha}^{\min}(\mathbf{v}) \subset {}_{a} \bigotimes_{\beta \in S(\alpha)} U_{\beta}^{\min}(\mathbf{v}) \subset {}_{a} \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_{\beta}$. Moreover, by Proposition 3, for $\beta \in S(\alpha)$ we have

$$\begin{split} U_{\beta}^{\min}(\mathbf{v}) &= \operatorname{span} \left\{ (id_{\beta} \otimes \varphi_{[\beta]})(\mathbf{v}_{\alpha}) : \mathbf{v}_{\alpha} \in U_{\alpha}^{\min}(\mathbf{v}) \text{ and } \varphi_{[\beta]} \in {}_{a} \bigotimes_{\delta \in S(\alpha) \setminus \{\beta\}} \mathbf{V}_{\delta}^{*} \right\} \\ &= \operatorname{span} \left\{ (id_{\beta} \otimes \varphi_{[\beta]}) \circ (id_{\alpha} \otimes \varphi_{[\alpha]})(\mathbf{v}) : \varphi_{[\alpha]} \in {}_{a} \bigotimes_{\mu \in S(D) \setminus \{\alpha\}} \mathbf{V}_{\mu}^{*} \text{ and } \varphi_{[\beta]} \in {}_{a} \bigotimes_{\delta \in S(\alpha) \setminus \{\beta\}} \mathbf{V}_{\delta}^{*} \right\}. \end{split}$$

Thus, $(id_{\alpha} \otimes \varphi_{[\alpha]})(\mathbf{v}) \in U_{\alpha}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha} \subset \mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}$, and hence

$$(id_{\beta}\otimes \boldsymbol{\varphi}_{[\beta]})\circ (id_{\alpha}\otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v})\in U_{\beta}^{\min}(\mathbf{v})\subset \mathbf{V}_{\beta}\subset \mathbf{V}_{\beta}_{\|\cdot\|_{\beta}}$$

when $\#\beta \geq 2$. However, if $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_{D}}} \setminus \mathbf{V}_{D}$ then $\overline{(id_{\alpha} \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) \in U_{\alpha}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}$. Since $\|\cdot\|_{\alpha} \gtrsim \|\cdot\|_{\vee(S(\alpha))}$ also by Lemma 2 we have $\overline{id_{\beta} \otimes \boldsymbol{\varphi}_{[\beta]}} \in \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}, \mathbf{V}_{\beta_{\|\cdot\|_{\beta}}})$. In consequence, a natural extension of the definition of minimal subspace $U_{\beta}^{\min}(\mathbf{v})$, for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_{D}}} \setminus \mathbf{V}_{D}$, is given by

$$U_{\beta}^{\min}(\mathbf{v}) := \operatorname{span}\left\{\overline{(id_{\beta} \otimes \varphi_{[\beta]})} \circ \overline{(id_{\alpha} \otimes \varphi_{[\alpha]})}(\mathbf{v}) : \varphi_{[\alpha]} \in {}_{a}\bigotimes_{\mu \in S(D) \setminus \{\alpha\}} \mathbf{V}_{\mu}^{*} \text{ and } \varphi_{[\beta]} \in {}_{a}\bigotimes_{\delta \in S(\alpha) \setminus \{\beta\}} \mathbf{V}_{\delta}^{*}\right\}.$$

To simplify the notation, we can write

$$\overline{(id_{\beta}\otimes\boldsymbol{\varphi}_{[\beta,\alpha]})}(\mathbf{v}):=\overline{(id_{\beta}\otimes\boldsymbol{\varphi}_{[\beta]})}\circ\overline{(id_{\alpha}\otimes\boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v})$$

where $\varphi_{[\beta,\alpha]} := \varphi_{[\alpha]} \otimes \varphi_{[\beta]} \in \left(a \bigotimes_{\mu \in S(D) \setminus \{\alpha\}} \mathbf{V}_{\mu}^{*} \right) \otimes_{a} \left(a \bigotimes_{\delta \in S(\alpha) \setminus \{\beta\}} \mathbf{V}_{\delta}^{*} \right)$ and $\overline{(id_{\beta} \otimes \varphi_{[\beta,\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_{D}}}, \mathbf{V}_{\beta_{\|\cdot\|_{\beta}}})$. Proceeding inductively, from the root to the leaves, we define the minimal subspace $U_{j}^{\min}(\mathbf{v})$ for each $j \in \mathcal{L}(T_{D})$ such that there exists $\eta \in T_{D} \setminus \{D\}$ with $j \in S(\eta)$ as

$$U_j^{\min}(\mathbf{v}) := \operatorname{span}\left\{\overline{(id_j \otimes \varphi_{[j,\eta,\dots,\beta,\alpha]})}(\mathbf{v}) : \varphi_{[j,\eta,\dots,\beta,\alpha]} \in \mathbf{W}_j\right\},\,$$

where

f

$$\mathbf{W}_j := \left(\begin{smallmatrix} a \\ \mu \in S(D) \setminus \{\alpha\} \end{smallmatrix} \right) \otimes_a \left(\begin{smallmatrix} a \\ \delta \in S(\alpha) \setminus \{\beta\} \end{smallmatrix} \right) \otimes_a \cdots \otimes_a \left(\begin{smallmatrix} a \\ \infty \\ k \in S(\eta) \setminus \{j\} \end{smallmatrix} \right)$$

With this extension the following result can be shown (see Lemma 3.13 in [3]).

Lemma 3 Let $\{\mathbf{V}_{\alpha_{\parallel},\parallel_{\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\parallel},\parallel_{D}} = \|\cdot\|_{D} \bigotimes_{j\in D} V_{j}$, in the topological tree-based format and assume that (17) holds. Let $\{\mathbf{v}_{n}\}_{n\geq 0} \subset \mathbf{V}_{D_{\parallel},\parallel_{D}}$ with $\mathbf{v}_{n} \rightharpoonup \mathbf{v}$, and $\mu \in T_{D} \setminus (\{D\} \cup \mathcal{L}(T_{D}))$. Then for each $\gamma \in S(\mu)$ we have

$$\overline{(id_{\gamma} \otimes \varphi_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}_{n}) \rightharpoonup \overline{(id_{\gamma} \otimes \varphi_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}) \text{ in } \mathbf{V}_{\gamma_{\parallel} \cdot \parallel_{\gamma}},$$

for all $\varphi_{[\gamma,\mu,\cdots,\beta,\alpha]} \in \left(a \bigotimes_{\mu \in S(D) \setminus \{\alpha\}} \mathbf{V}_{\mu}^{*} \right) \otimes_{a} \left(a \bigotimes_{\delta \in S(\alpha) \setminus \{\beta\}} \mathbf{V}_{\delta}^{*} \right) \otimes_{a} \cdots \otimes_{a} \mathcal{O}_{a} \otimes_{\eta \in S(\mu) \setminus \{\gamma\}} V_{\eta}^{*} \right).$

Then in a similar way as Theorem 3.15 in [3] the following theorem can be shown.

Theorem 4 Let $\{\mathbf{V}_{\alpha_{\parallel},\parallel_{\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\parallel},\parallel_{D}} = \underset{\parallel}{\parallel}_{\parallel} \bigotimes_{j\in D} V_{j}$, in the topological tree-based format and assume that (17) holds. Let $\{\mathbf{v}_{n}\}_{n\geq 0} \subset \mathbf{V}_{D_{\parallel},\parallel_{D}}$ with $\mathbf{v}_{n} \rightharpoonup \mathbf{v}$, then

$$\dim \overline{U_{\alpha}^{\min}(\mathbf{v})}^{\|\cdot\|_{\alpha}} = \dim U_{\alpha}^{\min}(\mathbf{v}) \le \liminf_{n \to \infty} \dim U_{\alpha}^{\min}(\mathbf{v}_n),$$

for all $\alpha \in T_D \setminus \{D\}$.

Now, following the proof of Theorem 4.1 in [3] we obtain the final theorem.

Theorem 5 Let $\mathbf{V}_D = {}_a \bigotimes_{j \in D} V_j$ and let $\{\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a reflexive Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D} \bigotimes_{j \in D} V_j$ in the topological tree-based format and assume that (17) holds. Then the set $\mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D)$ is weakly closed in $\mathbf{V}_{D_{\|\cdot\|_D}}$ and hence for each $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ there exists $\mathbf{u}_{best} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D)$ such that

$$\|\mathbf{v} - \mathbf{u}_{best}\|_{D} = \min_{\mathbf{u} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_{D})} \|\mathbf{v} - \mathbf{u}\|_{D}.$$

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