Geometry of tree-based tensor formats in tensor Banach spaces

Antonio Falcó*[1], Wolfgang Hackbusch[2] and Anthony Nouy[3]
[1] ESI International Chair@CEU-UCH,
Departamento de Matemáticas, Física y Ciencias Tecnológicas,
Universidad Cardenal Herrera-CEU, CEU Universities
San Bartolomé 55, 46115 Alfara del Patriarca (Valencia), Spain
e-mail: `afalco@uchceu.es`
[2] Max-Planck-Institut *Mathematik in den Naturwissenschaften*
Inselstr. 22, D-04103 Leipzig, Germany
e-mail: `wh@mis.mpg.de`
[3] Centrale Nantes,
LMJL UMR CNRS 6629
1 rue de la Noë, 44321 Nantes Cedex 3, France.
e-mail: `anthony.nouy@ec-nantes.fr`

Abstract

In the paper *'On the Dirac-Frenkel Variational Principle on Tensor Banach Spaces'*, we provided a geometrical description of manifolds of tensors in Tucker format with fixed multilinear (or Tucker) rank in tensor Banach spaces, that allowed to extend the Dirac-Frenkel variational principle in the framework of topological tensor spaces. The purpose of this note is to extend these results to more general tensor formats. More precisely, we provide a new geometrical description of manifolds of tensors in tree-based (or hierarchical) format, also known as tree tensor networks, which are intersections of manifolds of tensors in Tucker format associated with different partitions of the set of dimensions. The proposed geometrical description of tensors in tree-based format is compatible with the one of manifolds of tensors in Tucker format.

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*Corresponding author

1 Introduction

Tensor methods are prominent tools in a wide range of applications involving high-dimensional data or functions. The exploitation of low-rank structures of tensors is the basis of many approximation or dimension reduction methods, see the surveys [15, 2, 17, 18, 4, 5] and monograph [11]. Providing a geometrical description of sets of low-rank tensors has many interests. In particular, it allows to devise robust algorithms for optimization [1, 23] or construct reduced order models for dynamical systems [14].

A basic low-rank tensor format is the Tucker format. Given a collection of d vector spaces V_ν, $\nu \in D := \{1, \ldots, d\}$, and the corresponding algebraic tensor space $\mathbf{V}_D = V_1 \otimes \ldots \otimes V_d$, the set of tensors $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ of tensors in Tucker format with rank $\mathfrak{r} = (r_1, \ldots, r_d)$ is the set of tensors $\mathbf{v}$ in $\mathbf{V}_D$ such that $\mathbf{v} \in U_1 \otimes \ldots \otimes U_d$ for some subspaces U_ν in the Grassmann manifold $\mathbb{G}_{r_\nu}(V_\nu)$ of r_ν-dimensional spaces in V_ν. A geometrical description of $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ has been introduced in [9], providing this set the structure of a C^∞-Banach manifold.

Tree-based tensor formats [8], also known as tree tensor networks in physics or data science [19, 21, 10, 16], are more general low-rank tensor formats, also based on subspaces. They include the hierarchical format [12] or the tensor-train format [20]. Sets of tensors in tree-based tensor format are the intersection of a collection of sets of tensors in Tucker format associated with a hierarchy of partitions given by a tree. More precisely, given a dimension partition tree T_D over D, we can define a sequence of partitions $\mathcal{P}_1, \ldots, \mathcal{P}_L$ of D, with L the depth of the tree, such that each element in $\mathcal{P}_k$ is a subset of an element of $\mathcal{P}_{k-1}$ (see example in Figure 1.1). For each partition $\mathcal{P}_k$, a tensor in $\mathbf{V}_D$ can be identified with a tensor in $\mathbf{V}_{\mathcal{P}_k} := \bigotimes_{\alpha \in \mathcal{P}_k} \mathbf{V}_\alpha$, that allows to define manifolds of tensors in Tucker format $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_{\mathcal{P}_k})$ with $\mathfrak{r}_k \in \mathbb{N}^{\#\mathcal{P}_k}$. The set $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of tensors in $\mathbf{V}_D$ with tree-based rank $\mathfrak{r} = (r_\alpha)_{\alpha \in T_D} \in \mathbb{N}^{\#T_D}$ is then given by

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) = \bigcap_{k=1}^{L} \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_{\mathcal{P}_k})$$

where $\mathfrak{r}_k = (r_\alpha)_{\alpha \in \mathcal{P}_k}$.

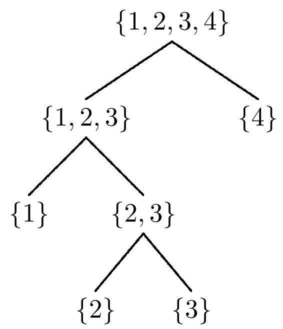

Figure 1.1: A dimension partition tree over $D = \{1, 2, 3, 4\}$, with depth $L = 3$, and the associated partitions of D: $\mathcal{P}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, $\mathcal{P}_2 = \{\{1\}, \{2, 3\}, \{4\}\}$, $\mathcal{P}_1 = \{\{1, 2, 3\}, \{4\}\}$.

In this paper, we provide a new geometrical description of the sets $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of tensors with fixed tree-based rank in tensor Banach spaces. This description is compatible with the one of manifolds $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_{\mathcal{P}_k})$ introduced in [9]. It is different from the ones from [22] and [13], respectively introduced for hierarchical and tensor train formats in finite-dimensional tensor spaces. It is also different from the one introduced by the authors in [7], that provided a different chart system. The present geometrical description is more natural and we believe that it is more amenable for applications. With the present description, and under similar assumptions on the norms of tensor spaces, Theorem 5.2 and Theorem 5.4 in [9] also hold for tree-based tensor formats, that allows to extend the Dirac-Frenkel variational principle for tree-based tensor formats in tensor Banach spaces.

The outline of this note is as follows. We start in section 2 by recalling results from [9], that is a geometrical description of manifolds $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ of tensors in Tucker format. Then in section 3 we introduce a description of tree-based tensor formats $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ as an intersection of Tucker formats. Finally in Section

4, we introduce the new geometrical description of the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of tensors in tree-based tensor format with fixed tree-based rank.

2 Preliminary results

Let $D := \{1, \ldots, d\}$ be a finite index set and consider an algebraic tensor space $\mathbf{V}_D = _a \bigotimes_{\alpha \in D} V_\alpha$ generated from vector spaces V_α, $\alpha \in D$ (the suffix 'a' in $_a \bigotimes_{\alpha \in D} V_\alpha$ refers to the 'algebraic' nature). For any partition $\mathcal{P}_D$ of D, the algebraic tensor space $\mathbf{V}_D$ can be identified with an algebraic tensor space generated from vector spaces $\mathbf{V}_\alpha$, $\alpha \in \mathcal{P}_D$. Indeed, for any partition $\mathcal{P}_D$ of D, the equality

$$\mathbf{V}_D = {}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_\alpha$$

holds, with $\mathbf{V}_\alpha := {}_a \bigotimes_{j \in \alpha} V_j$ if $\alpha \neq \{j\}$, for some $j \in D$, or $\mathbf{V}_\alpha = V_j$ if $\alpha = \{j\}$ for some $j \in D$. Next we identify D with the trivial partition $\{\{1\}, \{2\}, \ldots, \{d\}\}$.

Remark 2.1 *In [9], we considered the tensor space $\mathbf{V}_D = _a \bigotimes_{\alpha \in D} V_\alpha$ for a given D. It is not difficult to check that the results from [9] remain true when substituting D by any partition $\mathcal{P}_D$, that includes the initial case by identifying D with the trivial partition $\{\{1\}, \{2\}, \ldots, \{d\}\}$. More precisely, we can substitute with minor changes along the paper "$\alpha \in D$" by "$\alpha \in \mathcal{P}_D$".*

Before restating Theorem 3.17 of [9] in the present framework, we recall some definitions from [9].

Let X be a Banach space. We denote by $\mathbb{G}(X)$ the Grassmann manifold of closed subspaces in X (see Section 2 in [9]). More precisely, we say that $U \in \mathbb{G}(X)$ holds if and only if U is a closed subspace in X and there exists a closed subspace W in X such that $X = U \oplus W$. Every finite-dimensional subspace belongs to $\mathbb{G}(X)$, and we denote by $\mathbb{G}_n(X)$ the space of all n-dimensional subspaces of X $(n \geq 0)$.

Assume that $\mathcal{P}_D$ is a partition of D. Given $\mathfrak{r} = (r_\alpha)_{\alpha \in \mathcal{P}_D} \in \mathbb{N}^{\#\mathcal{P}_D}$, we denote by $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ the set of tensors in $\mathbf{V}_D$ represented in Tucker format with a fixed rank $\mathfrak{r}$. In particular, $\mathbf{v} \in \mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ if and only if for each $\alpha \in \mathcal{P}_D$ there exists a minimal subspace $\mathbf{U}_\alpha := \mathbf{U}_\alpha(\mathbf{v}) \in \mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha)$ such that $\mathbf{v} \in {}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{U}_\alpha$. Recall that the finite dimensional vector space ${}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{U}_\alpha$ is linearly isomorphic to the vector space $\mathbb{R}^{\times_{\alpha \in \mathcal{P}_D} r_\alpha}$. This fact allows to identify the set of full rank tensors in $\mathbb{R}^{\times_{\alpha \in \mathcal{P}_D} r_\alpha}$, denoted by $\mathbb{R}_*^{\times_{\alpha \in \mathcal{P}_D} r_\alpha}$, with $\mathfrak{M}_{\mathfrak{r}}\left({}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{U}_\alpha\right)$.

Then we have the following result.

Theorem 2.2 *Assume that $\mathcal{P}_D$ is a partition of D, $(\mathbf{V}_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in \mathcal{P}_D$ and that $\|\cdot\|_D$ is a norm on the tensor space $\mathbf{V}_D = {}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_\alpha$ such that the tensor product map*

$$\bigotimes_{\alpha \in \mathcal{P}_D} : \left(\bigtimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_\alpha, \|\cdot\|_\times \right) \longrightarrow \left({}_a \bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_\alpha \, , \|\cdot\|_D \right), \tag{2.1}$$

is continuous. Then there exists a $\mathcal{C}^\infty$-atlas $\{\mathcal{U}(\mathbf{v}), \xi_{\mathbf{v}}\}_{\mathbf{v} \in \mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)}$ for $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ and hence $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ is a $\mathcal{C}^\infty$-Banach manifold modelled on a Banach space

$$\left(\bigtimes_{\alpha \in \mathcal{P}_D} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha) \right) \times \mathbb{R}^{\times_{\alpha \in \mathcal{P}_D} r_\alpha}.$$

Here $\mathbf{U}_\alpha \in \mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha)$ and $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha \oplus \mathbf{W}_\alpha$, where $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is the completion of $\mathbf{V}_\alpha$ for $\alpha \in \mathcal{P}_D$. Moreover,

$$\left(\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D), \bigtimes_{\alpha \in \mathcal{P}_D} \mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha), \varrho_{\mathfrak{r}} \right)$$

is a $\mathcal{C}^\infty$-fibre bundle with typical fibre $\mathbb{R}_*^{\times_{\alpha \in \mathcal{P}_D} r_\alpha}$, with $\varrho_{\mathfrak{r}}$ being the map which associates to a tensor $\mathbf{v}$ in $\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ its minimal subspaces $\mathbf{U}_\alpha$, $\alpha \in \mathcal{P}_D$.

In (2.1) the product space $\times_{\alpha\in\mathcal{P}_D}\mathbf{V}_\alpha$ is equipped with the product topology induced by the maximum norm $\|(\mathbf{v}_\alpha)_{\alpha\in\mathcal{P}_D})\|_\times = \max_{\alpha\in\mathcal{P}_D}\|\mathbf{v}_\alpha\|_\alpha$.

We recall that $\overline{\mathbf{V}_D}^{\|\cdot\|_D} = \mathbf{V}_{D_{\|\cdot\|_D}}$ denotes the tensor Banach space obtained as the completion of the algebraic one

$$\mathbf{V}_D = {}_a\bigotimes_{\alpha\in\mathcal{P}_D}\mathbf{V}_\alpha$$

under the norm $\|\cdot\|_D$. Our next step is, given a fixed partition $\mathcal{P}_D$ of D, to identify the Banach space $\times_{\alpha\in\mathcal{P}_D}\mathcal{L}(\mathbf{U}_\alpha,\mathbf{W}_\alpha)$ with a closed subspace of the Banach algebra $\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})$. To this end, we need to proceed in the framework of Section 4 in [9]. First, we recall the definition of injective norm (Definition 4.9 in [9]) stated in the actual framework.

Definition 2.3 *Let $\mathbf{V}_\alpha$ be a Banach space with norm $\|\cdot\|_\alpha$ for $\alpha\in\mathcal{P}_D$. Then for $\mathbf{v}\in\mathbf{V} = {}_a\bigotimes_{\alpha\in\mathcal{P}_D}\mathbf{V}_\alpha$ define $\|\cdot\|_{\vee((\mathbf{V}_\alpha)_{\alpha\in\mathcal{P}_D})}$ by*

$$\|\mathbf{v}\|_{\vee((\mathbf{V}_\alpha)_{\alpha\in\mathcal{P}_D})} := \sup\left\{\frac{\left|\left(\bigotimes_{\alpha\in\mathcal{P}_D}\varphi_\alpha\right)(\mathbf{v})\right|}{\prod_{\alpha\in\mathcal{P}_D}\|\varphi_\alpha\|_\alpha^*} : 0\neq\varphi_\alpha\in\mathbf{V}_\alpha^*, \alpha\in\mathcal{P}_D\right\}, \tag{2.2}$$

where $\mathbf{V}_\alpha^*$ is the continuous dual of $\mathbf{V}_\alpha$.

Let W and U be closed subspaces of a Banach space X such that $X = U\oplus W$. From now on, we will denote by $P_{U\oplus W}$ the projection onto U along W. Then we have $P_{W\oplus U} = id_X - P_{U\oplus W}$. The proof of the next result uses Proposition 2.8, Lemma 4.13 and Lemma 4.14 in [9].

Lemma 2.4 *Assume that $(\mathbf{V}_\alpha,\|\cdot\|_\alpha)$ is a normed space for each $\alpha\in\mathcal{P}_D$ and let $\|\cdot\|_D$ be a norm on the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha\in\mathcal{P}_D}\mathbf{V}_\alpha$ such that*

$$\|\cdot\|_{\vee((\mathbf{V}_\alpha)_{\alpha\in\mathcal{P}_D})} \lesssim \|\cdot\|_D\,, \tag{2.3}$$

holds. Let $\mathbf{U}_\alpha\in\mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha)$ and $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha\oplus\mathbf{W}_\alpha$, where $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is the completion of $\mathbf{V}_\alpha$ for $\alpha\in\mathcal{P}_D$. Then for each $\alpha\in\mathcal{P}_D$ we have

$$\mathcal{L}(\mathbf{U}_\alpha,\mathbf{W}_\alpha)\otimes_a\operatorname{span}\{\mathbf{id}_{[\alpha]}\}\in\mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})\right)$$

where $\mathbf{id}_{[\alpha]} := \bigotimes_{\beta\in\mathcal{P}_D\setminus\{\alpha\}} id_{\mathbf{V}_\beta}$. Furthermore,

$$\bigoplus_{\alpha\in\mathcal{P}_D}\mathcal{L}(\mathbf{U}_\alpha,\mathbf{W}_\alpha)\otimes_a\operatorname{span}\{\mathbf{id}_{[\alpha]}\}\in\mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})\right).$$

Proof. To prove the lemma, for a fixed $\alpha\in\mathcal{P}_D$, note that $id_{\mathbf{V}_\alpha} = P_{\mathbf{U}_\alpha\oplus\mathbf{W}_\alpha} + P_{\mathbf{W}_\alpha\oplus\mathbf{U}_\alpha}$ and write

$$id_{\mathbf{V}_{D_{\|\cdot\|_D}}} = id_{\mathbf{V}_\alpha}\otimes\mathbf{id}_{[\alpha]}.$$

Clearly, $P_{\mathbf{U}_\alpha\oplus\mathbf{W}_\alpha}\otimes\mathbf{id}_{[\alpha]}\in\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})$. Then by proceeding as in Lemma 4.13 in [9] we obtain that

$$P_{\mathbf{W}_\alpha\oplus\mathbf{U}_\alpha}\otimes\mathbf{id}_{[\alpha]}\in\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}}).$$

Now, define the linear and bounded map

$$\mathcal{P}_\alpha : \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})\longrightarrow\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})$$

as $\mathcal{P}_\alpha(L) = (P_{\mathbf{W}_\alpha\oplus\mathbf{U}_\alpha}\otimes\mathbf{id}_{[\alpha]})\circ L\circ(P_{\mathbf{U}_\alpha\oplus\mathbf{W}_\alpha}\otimes\mathbf{id}_{[\alpha]})$. It satisfies $\mathcal{P}_\alpha\circ\mathcal{P}_\alpha = \mathcal{P}_\alpha$ and

$$\mathcal{P}_\alpha(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}},\mathbf{V}_{D_{\|\cdot\|_D}})) = \mathcal{L}(\mathbf{U}_\alpha,\mathbf{W}_\alpha)\otimes_a\operatorname{span}\{\mathbf{id}_{[\alpha]}\}.$$

Proposition 2.8(b) in [9] implies that $\mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\} \in \mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{D_{\|\cdot\|_D}})\right)$. Observe that for $\alpha, \beta \in \mathcal{P}_D$ with $\alpha \neq \beta$ we have

$$(\mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\}) \cap (\mathcal{L}\left(\mathbf{U}_\beta, \mathbf{W}_\beta\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\beta]}\}) = \{\mathbf{0}\}.$$

By Lemma 4.14 in [9] we have

$$\bigoplus_{\alpha \in \mathcal{P}_D} \mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\} \in \mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{D_{\|\cdot\|_D}})\right).$$

This proves the lemma. ■

Lemma 2.4 allows to introduce the following linear isomorphism:

$$\Delta : \bigtimes_{\alpha \in \mathcal{P}_D} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha) \longrightarrow \bigoplus_{\alpha \in \mathcal{P}_D} \mathcal{L}\left(U_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\}, \quad (L_\alpha)_{\alpha \in \mathcal{P}_D} \mapsto \sum_{\alpha \in \mathcal{P}_D} L_\alpha \otimes \mathbf{id}_{[\alpha]}.$$

where $\mathbf{id}_{[\alpha]} := \bigotimes_{\beta \in \mathcal{P}_D \setminus \{\alpha\}} id_{\mathbf{V}_\beta}$ for $\alpha \in \mathcal{P}_D$. The next proposition gives us a useful property of the elements in the image of the map Δ.

Proposition 2.5 *Assume that $\mathcal{P}_D$ is a partition of D, $(\mathbf{V}_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in \mathcal{P}_D$ and $\|\cdot\|_D$ is a norm on the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in \mathcal{P}_D} \mathbf{V}_\alpha$ such that (2.3) holds. Then for each $(L_\alpha)_{\alpha \in \mathcal{P}_D} \in \bigtimes_{\alpha \in \mathcal{P}_D} \mathcal{L}(U_\alpha, \mathbf{W}_\alpha)$ it holds that*

$$\exp(\Delta\left((L_\alpha)_{\alpha \in \mathcal{P}_D}\right)) = \bigotimes_{\alpha \in \mathcal{P}_D} \exp(L_\alpha).$$

Proof. Put $L := \Delta\left((L_\alpha)_{\alpha \in \mathcal{P}_D}\right) = \sum_{\alpha \in \mathcal{P}_D} L_\alpha \otimes \mathbf{id}_{[\alpha]}$ and observe that for each $\alpha \in \mathcal{P}_D$ it holds

$$\exp(L_\alpha \otimes \mathbf{id}_{[\alpha]}) = \sum_{n=0}^{\infty} \frac{1}{n!}(L_\alpha \otimes \mathbf{id}_{[\alpha]})^n = \left(\sum_{n=0}^{\infty} \frac{1}{n!} L_\alpha^n\right) \otimes \mathbf{id}_{[\alpha]} = \exp(L_\alpha) \otimes \mathbf{id}_{[\alpha]}.$$

Moreover for $\alpha, \beta \in \mathcal{P}_D$ and $\alpha \neq \beta$ we have

$$(L_\alpha \otimes \mathbf{id}_{[\alpha]}) \circ (L_\beta \otimes \mathbf{id}_{[\beta]}) = (L_\beta \otimes \mathbf{id}_{[\beta]}) \circ (L_\alpha \otimes \mathbf{id}_{[\alpha]}) = L_\alpha \otimes L_\beta \otimes \left(\bigotimes_{\delta \in \mathcal{P}_D \setminus \{\alpha, \beta\}} id_{\mathbf{V}_\delta}\right).$$

Finally, by seing $\mathcal{P}_D$ as an ordered set, and by denoting $\bigodot_{i=1}^n A_i := A_1 \circ A_2 \circ \cdots \circ A_n$ is the composition of maps A_i, $1 \leq i \leq n$, we have

$$\exp\left(L\right) = \bigodot_{\alpha \in \mathcal{P}_D} \exp(L_\alpha \otimes \mathbf{id}_{[\alpha]}) = \bigodot_{\alpha \in \mathcal{P}_D} \exp(L_\alpha) \otimes \mathbf{id}_{[\alpha]} = \bigotimes_{\alpha \in \mathcal{P}_D} \exp(L_\alpha).$$

Note that since operators $\exp(L_\alpha) \otimes \mathbf{id}_{[\alpha]}$ and $\exp(L_\beta) \otimes \mathbf{id}_{[\beta]}$ commute for any $\alpha, \beta \in \mathcal{P}_D$, the above result is independent of the chosen order on $\mathcal{P}_D$. This proves the proposition. ■

To conclude we re-state Theorem 2.2 as follows.

Theorem 2.6 *Assume that $\mathcal{P}_D$ is a partition of D, $(V_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in \mathcal{P}_D$ and let $\|\cdot\|_D$ be a norm on the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in D} \mathbf{V}_\alpha$ such that* (2.3) *holds. Then there exists a $\mathcal{C}^\infty$-atlas $\{\mathcal{U}(\mathbf{v}), \xi_\mathbf{v}\}_{\mathbf{v} \in \mathfrak{M}_\mathfrak{r}(\mathbf{V}_D)}$ for $\mathfrak{M}_\mathfrak{r}(\mathbf{V}_D)$ and hence $\mathfrak{M}_\mathfrak{r}(\mathbf{V}_D)$ is a $\mathcal{C}^\infty$-Banach manifold modelled on a Banach space*

$$\left(\bigoplus_{\alpha \in \mathcal{P}_D} \mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\}\right) \times \mathbb{R}^{\bigtimes_{\alpha \in \mathcal{P}_D} r_\alpha},$$

here $\mathbf{U}_\alpha \in \mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha)$ and $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha \oplus \mathbf{W}_\alpha$, where $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is the completion of $\mathbf{V}_\alpha$ for $\alpha \in \mathcal{P}_D$. Moreover,

$$\left(\mathfrak{M}_\mathfrak{r}(\mathbf{V}_D), \bigtimes_{\alpha \in \mathcal{P}_D} \mathbb{G}_{r_\alpha}(\mathbf{V}_\alpha), \varrho_\mathfrak{r}\right)$$

is a $\mathcal{C}^\infty$-fibre bundle with typical fibre $\mathbb{R}_*^{\bigtimes_{\alpha \in \mathcal{P}_D} r_\alpha}$.

We recall the construction of the local charts $\xi_{\mathbf{v}}$ for a given $\mathbf{v} \in \mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$. Once minimal subspaces $\mathbf{U}_\alpha$ of $\mathbf{v} \in {}_a\bigotimes_{\alpha\in\mathcal{P}_D} \mathbf{U}_\alpha$ are obtained for each $\alpha \in \mathcal{P}_D$, we can choose closed subspaces $\mathbf{W}_\alpha$ in $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ such that $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha \oplus \mathbf{W}_\alpha$ holds. Then $\xi_{\mathbf{v}}^{-1}$ is a bijection between the set

$$\left(\bigoplus_{\alpha\in\mathcal{P}_D} \mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\}\right) \times \mathfrak{M}_{\mathfrak{r}_k}\left({}_a\bigotimes_{\alpha\in\mathcal{P}_D} \mathbf{U}_\alpha\right)$$

and the subset $\mathcal{U}(\mathbf{v}) \subset \mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D)$ containing $\mathbf{v}$. To simplify notation, let

$$\mathbf{E}_{\mathcal{P}_D}(\mathbf{v}) := \left(\bigoplus_{\alpha\in\mathcal{P}_D} \mathcal{L}\left(\mathbf{U}_\alpha, \mathbf{W}_\alpha\right) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\}\right).$$

It can be shown (see Lemma 3.12 in [9]) that $\mathbf{w} \in \mathcal{U}(\mathbf{v})$ if and only if there exists a unique

$$(\Delta((L_\alpha)_{\alpha\in\mathcal{P}_D}), \mathbf{u}) \in \mathbf{E}_{\mathcal{P}_D}(\mathbf{v}) \times \mathfrak{M}_{\mathfrak{r}_k}\left({}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right)$$

such that

$$\mathbf{w} = \xi_{\mathbf{v}}^{-1}(\Delta(L_\alpha)_{\alpha\in\mathcal{P}_D}, \mathbf{u}) = \left(\bigotimes_{\alpha\in\mathcal{P}_D} \exp(L_\alpha)\right)(\mathbf{u}). \tag{2.4}$$

Put $L := \Delta((L_\alpha)_{\alpha\in\mathcal{P}_D})$, thanks to Proposition 2.5, the equality (2.4) is equivalent to

$$\mathbf{w} = \xi_{\mathbf{v}}^{-1}(L, \mathbf{u}) = \exp(L)(\mathbf{u}), \tag{2.5}$$

where $L = \sum_{\alpha\in\mathcal{P}_D} L_\alpha \otimes \mathbf{id}_{[\alpha]}$ is a Laplacian-like map. Thus, every tensor in Tucker format is locally characterised by a full-rank tensor and a Laplacian-like map.

3 The set of tensors in tree-based format with fixed tree-based rank

To introduce the set of tensors in tree-based format with fixed tree-based rank we shall use the minimal subspaces, in particular, Proposition 2.6 in [9] (see also [6] or [11]). Let $\mathcal{P}_D$ be a given partition of D. By definition of the minimal subspaces $U_\alpha^{\min}(\mathbf{v})$, $\alpha \in \mathcal{P}_D$, we have

$$\mathbf{v} \in {}_a\bigotimes_{\alpha\in\mathcal{P}_D} U_\alpha^{\min}(\mathbf{v}) \ .$$

For a given $\alpha \in \mathcal{P}_D$ with $\#\alpha \geq 2$ and any partition $\mathcal{P}_\alpha$ of α, we also have

$$\mathbf{v} \in \left({}_a\bigotimes_{\beta\in\mathcal{P}_\alpha} U_\beta^{\min}(\mathbf{v})\right) \otimes_a \left({}_a\bigotimes_{\delta\in\mathcal{P}_D\setminus\{\alpha\}} U_\delta^{\min}(\mathbf{v})\right).$$

We finally recall a useful result on the relation between minimal subspaces (see Section 2 in [6]).

Proposition 3.1 *For any $\alpha \in 2^D$ with $\#\alpha \geq 2$ and any partition $\mathcal{P}_\alpha$ of α, it holds*

$$U_\alpha^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in\mathcal{P}_\alpha} U_\beta^{\min}(\mathbf{v}) \ .$$

In order to define tree-based tensor format we introduce the three definitions.

Definition 3.2 (Dimension partition tree) *A tree T_D is called* a dimension partition tree over D *if*

(a) *all vertices* $\alpha \in T_D$ *are non-empty subsets of* D,

(b) D *is the* root *of* T_D,

(c) *every vertex* $\alpha \in T_D$ *with* $\#\alpha \geq 2$ *has at least two sons and the set of sons of* α*, denoted* $S(\alpha)$*, is a non-trivial partition of* α,

(d) *every vertex* $\alpha \in T_D$ *with* $\#\alpha = 1$ *has no son and is called a* leaf.

The set of leaves is denoted by $\mathcal{L}(T_D)$.

A straightforward consequence of Definition 3.2 is that the set of leaves $\mathcal{L}(T_D)$ coincides with the singletons of D, i.e., $\mathcal{L}(T_D) = \{\{j\} : j \in D\}$.

Definition 3.3 (Levels, depth and partitions) *The* levels *of the vertices of a dimension partition tree* T_D*, denoted by* $\mathrm{level}(\alpha)$, $\alpha \in T_D$*, are integers defined such that* $\mathrm{level}(D) = 0$ *and for any pair* $\alpha, \beta \in T_D$ *such that* $\beta \in S(\alpha)$, $\mathrm{level}(\beta) = \mathrm{level}(\alpha)+1$*. The* depth[1] *of the tree* T_D *is defined as* $\mathrm{depth}(T_D) = \max_{\alpha\in T_D} \mathrm{level}(\alpha)$*. Then to each level* k *of* T_D*,* $1 \leq k \leq \mathrm{depth}(T_D)$*, is associated a partition of* D :

$$\mathcal{P}_k(T_D) = \{\alpha \in T_D : \mathrm{level}(\alpha) = k\} \cup \{\alpha \in \mathcal{L}(T_D) : \mathrm{level}(\alpha) < k\}.$$

Remark 3.4 *Note that for any tree,* $\mathcal{P}_1(T_D) = S(D)$ *and* $\mathcal{P}_{\mathrm{depth}(T_D)}(T_D) = \mathcal{L}(T_D)$*. Also note that some of the leaves of* T_D *may be contained in several partitions, and if* $\alpha \in \mathcal{L}(T_D)$*, then* $\alpha \in \mathcal{P}_k(T_D)$ *for* $\mathrm{level}(\alpha) \leq k \leq \mathrm{depth}(T_D)$.

For any partition $\mathcal{P}_k(T_D)$ of level k, $1 \leq k \leq \mathrm{depth}(T_D)$, we use the identification

$$\mathbf{V}_D = {}_a \bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{V}_\alpha \, .$$

This leads us to the following definition of the representation of the tensor space $\mathbf{V}_D$ in tree-based format.

Definition 3.5 *For a tensor space* $\mathbf{V}_D$ *and a dimension partition tree* T_D*, the pair* $(\mathbf{V}_D, T_D)$ *is called a representation of the tensor space* $\mathbf{V}_D$ *in* tree-based format*, and corresponds to the identification of* $\mathbf{V}_D$ *with tensor spaces* ${}_a \bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{V}_\alpha$ *of different levels* k*,* $1 \leq k \leq \mathrm{depth}(T_D)$.

Remark 3.6 *By Proposition 3.1, for each* $\mathbf{v} \in \mathbf{V}_D$*, it holds that*

$$\mathbf{v} \in {}_a \bigotimes_{\alpha\in\mathcal{P}_1(T_D)} U_\alpha^{\min}(\mathbf{v}) \subset {}_a \bigotimes_{\alpha\in\mathcal{P}_2(T_D)} U_\alpha^{\min}(\mathbf{v}) \subset \cdots \subset {}_a \bigotimes_{\alpha\in\mathcal{P}_{\mathrm{depth}(T_D)}(T_D)} U_\alpha^{\min}(\mathbf{v}) \, .$$

Example 3.7 (Tucker format) *In Figure 3.1,* $D = \{1, 2, 3, 4, 5, 6\}$ *and*

$$T_D = \{D, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

Here $\mathrm{depth}(T_D) = 1$ *and* $\mathcal{P}_1(T_D) = \mathcal{L}(T_D)$*. This tree is related to the basic identification of* $\mathbf{V}_D$ *with* ${}_a \bigotimes_{j=1}^{6} V_j$.

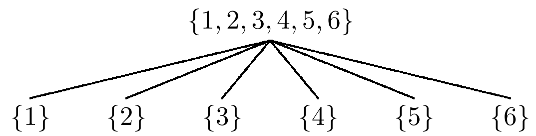

Figure 3.1: a dimension partition tree with $\mathrm{depth}(T_D) = 1$ (Tucker tree).

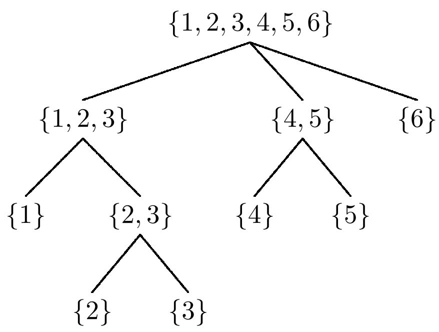

Figure 3.2: A dimension partition tree with $\operatorname{depth}(T_D) = 3$.

Example 3.8 *In Figure 3.2,* $D = \{1,2,3,4,5,6\}$ *and*

$$T_D = \{D, \{1,2,3\}, \{4,5\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

Here $\operatorname{depth}(T_D) = 3$, $\mathcal{P}_1(T_D) = \{\{1,2,3\}, \{4,5\}, \{6\}\}$, $\mathcal{P}_2(T_D) = \{\{1\}, \{2,3\}, \{4\}, \{5\}, \{6\}\}$ *and* $\mathcal{P}_3(T_D) = \mathcal{L}(T_D)$. *This tree is related to the identification of* $\mathbf{V}_D$ *with* ${}_a\bigotimes_{j=1}^{6} V_j$, $\mathbf{V}_D = V_1 \otimes_a \mathbf{V}_{23} \otimes_a V_4 \otimes_a V_5 \otimes_a V_6$ *and* $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For each $\mathbf{v} \in \mathbf{V}_D$, we have that $(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha \in 2^D \setminus \{\emptyset\}}$ is in $\mathbb{N}_0^{2^{\#D}-1}$.

Definition 3.9 (Tree-based rank) *For a given dimension partition tree* T_D *over* D*, we define the* tree-based rank *of a tensor* $\mathbf{v} \in \mathbf{V}_D$ *by the tuple* $\operatorname{rank}_{T_D}(\mathbf{v}) := (\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha \in T_D} \in \mathbb{N}_0^{\#T_D}$.

Definition 3.10 (Admissible ranks) *A tuple* $\mathfrak{r} := (r_\alpha)_{\alpha \in T_D} \in \mathbb{N}^{\#T_D}$ *is said to be an* admissible tuple for T_D *if there exists* $\mathbf{v} \in \mathbf{V}_D$ *such that* $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$ *for all* $\alpha \in T_D$. *The set of admissible ranks for the representation* $(\mathbf{V}_D, T_D)$ *of the tensor space* $\mathbf{V}_D$ *is denoted by*

$$\mathcal{AD}(\mathbf{V}_D, T_D) := \{(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha \in T_D} : \mathbf{v} \in \mathbf{V}_D\}.$$

Definition 3.11 *Let* T_D *be a given dimension partition tree and fix some tuple* $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$. *Then* the set of tensors of fixed tree-based rank $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) := \left\{\mathbf{v} \in \mathbf{V}_D : \dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for all } \alpha \in T_D\right\} \tag{3.1}$$

and the set of tensors of tree-based rank bounded by $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D, T_D) := \left\{\mathbf{v} \in \mathbf{V}_D : \dim U_\alpha^{\min}(\mathbf{v}) \leq r_\alpha \text{ for all } \alpha \in T_D\right\}. \tag{3.2}$$

For $\mathfrak{r}, \mathfrak{s} \in \mathbb{N}_0^{\#T_D}$ we write $\mathfrak{s} \leq \mathfrak{r}$ if and only if $s_\alpha \leq r_\alpha$ for all $\alpha \in T_D$. Then for a fixed $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$, we have

$$\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D, T_D) := \bigcup_{\substack{\mathfrak{s} \leq \mathfrak{r} \\ \mathfrak{s} \in \mathcal{AD}(\mathbf{V}_D, T_D)}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D, T_D). \tag{3.3}$$

For each partition $\mathcal{P}_k(T_D)$ of D, $1 \leq k \leq \operatorname{depth}(T_D)$, we can introduce a set of tensors in Tucker format with fixed rank $\mathfrak{r}_k := (r_\alpha)_{\alpha \in \mathcal{P}_k(T_D)}$ given by

$$\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, \mathcal{P}_k(T_D)) = \{\mathbf{v} \in \mathbf{V}_D : \dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for } \alpha \in \mathcal{P}_k(T_D)\}.$$

Theorem 3.12 *For a dimension partition tree* T_D *and for* $\mathfrak{r} = (r_\alpha)_{\alpha \in T_D} \in \mathcal{AD}(\mathbf{V}_D, T_D)$,

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) = \bigcap_{k=1}^{\operatorname{depth}(T_D)} \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, \mathcal{P}_k(T_D)).$$

[1]By using the notion of edge, that is, the connection between one vertex to another, then our definition of depth coincides with the classical definition of height, i.e. the longest downward path between the root and a leaf.

Remark 3.13 *We point out that in [9] we introduce a representation of* $\mathbf{V}_D$ *in Tucker format. Letting* T_D^{Tucker} *be the Tucker dimension partition tree (see example 3.7) and given* $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D^{Tucker})$*, the set of tensors with fixed Tucker rank* $\mathfrak{r}$ *is defined by*

$$\mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D) := \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D^{Tucker}) = \left\{ \mathbf{v} \in \mathbf{V}_D : \dim U_k^{\min}(\mathbf{v}) = r_k,\ k \in \mathcal{L}(T_D^{Tucker}) \right\}.$$

This leads to the following representation of $\mathbf{V}_D$ *in Tucker format:*

$$\mathbf{V}_D = \bigcup_{\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D^{Tucker})} \mathfrak{M}_{\mathfrak{r}}(\mathbf{V}_D).$$

Note that for any tree T_D *with* $\operatorname{depth}(T_D) = 1$*,*

$$\mathfrak{M}_{\mathfrak{r}_{\operatorname{depth}(T_D)}}(\mathbf{V}_D, \mathcal{P}_{\operatorname{depth}(T_D)}(T_D)) = \mathfrak{M}_{\mathfrak{r}_{\operatorname{depth}(T_D)}}(\mathbf{V}_D).$$

Finally, we need to take into account the following situation. Let T_D be the rooted tree given in Figure 3.3. For this rooted tree we have $\operatorname{depth}(T_D) = 2$ and also

$$\mathcal{P}_1(T_D) = \{\{1\}, \{2,3,4,5,6\}\},$$
$$\mathcal{P}_2(T_D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

From Lemma 2.4 in [6] it can be shown that $\dim U_{\{1\}}^{\min}(\mathbf{v}) = \dim U_{\{2,3,4,5,6\}}^{\min}(\mathbf{v})$ holds for all $\mathbf{v} \in \mathbf{V}_D$. Hence

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) = \mathfrak{M}_{\mathfrak{r}_1}(\mathbf{V}_D, \mathcal{P}_1(T_D)) \cap \mathfrak{M}_{\mathfrak{r}_2}(\mathbf{V}_D, \mathcal{P}_2(T_D)) = \mathfrak{M}_{\mathfrak{r}_2}(\mathbf{V}_D, \mathcal{P}_2(T_D))$$

holds because

$$\mathfrak{M}_{\mathfrak{r}_1}(\mathbf{V}_D, \mathcal{P}_1(T_D)) = \{\mathbf{v} \in \mathbf{V}_D : \dim U_{\{1\}}^{\min}(\mathbf{v}) = r_{\{1\}} = \dim U_{\{2,3,4,5,6\}}^{\min}(\mathbf{v})\}$$

contains

$$\mathfrak{M}_{\mathfrak{r}_2}(\mathbf{V}_D, \mathcal{P}_2(T_D)) = \{\mathbf{v} \in \mathbf{V}_D : \dim U_{\{i\}}^{\min}(\mathbf{v}) = r_{\{i\}},\ 1 \leq i \leq 6\}.$$

Thus in order to avoid this situation we introduce the following definition.

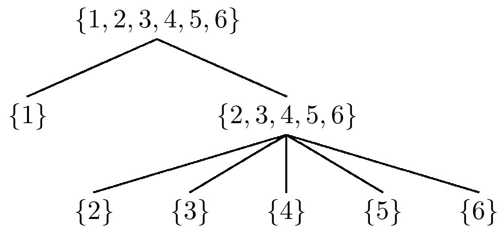

Figure 3.3: A dimension partition tree with $\operatorname{depth}(T_D) = 2$.

Definition 3.14 *For a dimension partition tree* T_D *and for* $\mathfrak{r} = (r_\alpha)_{\alpha \in T_D} \in \mathcal{AD}(\mathbf{V}_D, T_D)$*, we will say that* $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ *is a proper set of tree-based tensors with a fixed tree-based rank* $\mathfrak{r}$ *if*

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) \neq \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, \mathcal{P}_k(T_D)) \text{ holds for } 1 \leq k \leq \operatorname{depth}(T_D).$$

4 The geometry of tree-based tensor format

For a dimension partition tree T_D and for $\mathfrak{r} = (r_\alpha)_{\alpha \in T_D} \in \mathcal{AD}(\mathbf{V}_D, T_D)$, assume that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is a proper set of tree-based tensors with a fixed tree-based rank $\mathfrak{r}$ such that

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) = \bigcap_{k=1}^{\operatorname{depth}(T_D)} \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, \mathcal{P}_k(T_D)).$$

Assume that $(\mathbf{V}_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in \mathcal{P}_k(T_D)$ and that $\|\cdot\|_D$ is a norm on the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{V}_\alpha$ such that (2.3) holds for $1 \le k \le \mathrm{depth}(T_D)$.

From Theorem 2.6 we have that for each $1 \le k \le \mathrm{depth}(T_D)$ the collection $\mathcal{A}_k = \{(\mathcal{U}^{(k)}(\mathbf{v}), \xi_{\mathbf{v}}^{(k)})\}_{\mathbf{v}\in\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D,T_D)}$ is a $\mathcal{C}^\infty$-atlas for $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ and hence $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ is a $\mathcal{C}^\infty$-Banach manifold modelled on

$$\left(\bigoplus_{\alpha\in\mathcal{P}_k(T_D)} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha) \otimes_a \mathrm{span}\{\mathbf{id}_{[\alpha]}\}\right) \times \mathbb{R}^{\times_{\alpha\in\mathcal{P}_k(T_D)} r_\alpha},$$

where $\mathbf{U}_\alpha = U_\alpha^{\min}(\mathbf{v})$ is a r_α-dimensional subspace of $\mathbf{V}_\alpha$ for each $\alpha \in \mathcal{P}_k(T_D)$ where $\mathbf{v} \in {}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{U}_\alpha$ and $\mathbf{W}_\alpha$ is a closed subspace of $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ such that $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha \oplus \mathbf{W}_\alpha$, where $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is the completion of $\mathbf{V}_\alpha$ for $\alpha \in \mathcal{P}_k(T_D)$.

To simplify notation, here we write

$$\mathbf{E}_k(\mathbf{v}) := \left(\bigoplus_{\alpha\in\mathcal{P}_k(T_D)} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha) \otimes_a \mathrm{span}\{\mathbf{id}_{[\alpha]}\}\right)$$

for $1 \le k \le \mathrm{depth}(T_D)$. Next, we characterise the elements in the set

$$\bigcap_{k=1}^{\mathrm{depth}(T_D)} \xi_{\mathbf{v}}^{(k)}(\mathcal{U}^{(k)}(\mathbf{v})) = \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathbf{E}_k(\mathbf{v}) \times \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathfrak{M}_{\mathbf{r}_k}\left({}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right).$$

By using Remark 3.6 we have

$$\bigcap_{k=1}^{\mathrm{depth}(T_D)} {}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{U}_\alpha = \bigcap_{k=1}^{\mathrm{depth}(T_D)} {}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} U_\alpha^{\min}(\mathbf{v}) = {}_a\bigotimes_{\delta\in\mathcal{P}_1(T_D)} U_\delta^{\min}(\mathbf{v}) = {}_a\bigotimes_{\delta\in\mathcal{P}_1(T_D)} \mathbf{U}_\delta\,.$$

It allows us to take into account the open set

$$\mathcal{O}(\mathbf{v}) := \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathfrak{M}_{\mathfrak{r}_k}\left({}_a\bigotimes_{\alpha\in\mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right) \subset \left({}_a\bigotimes_{\delta\in\mathcal{P}_1(T_D)} \mathbf{U}_\delta\right) \bigcap \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D), \tag{4.1}$$

in ${}_a\bigotimes_{\delta\in\mathcal{P}_1(T_D)} U_\delta^{\min}(\mathbf{v})$, which is the smaller tensor product of minimal subspaces that contains $\mathbf{v}$.

Now,

$$\mathbf{E}(\mathbf{v}) := \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathbf{E}_k(\mathbf{v})$$

is a linear closed subspace in the Banach space $\mathbf{E}_k(\mathbf{v})$ for $1 \le k \le \mathrm{depth}(T_D)$ and also in $\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{D_{\|\cdot\|_D}})$. Hence $\mathbf{E}(\mathbf{v})$ is a Banach space. Moreover, given $L \in \mathbf{E}(\mathbf{v})$, for each $1 \le k \le \mathrm{depth}(T)$ there exists a unique

$$(L_\alpha^{(k)})_{\alpha\in\mathcal{P}_k(T_D)} \in \bigtimes_{\alpha\in\mathcal{P}_k(T_D)} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha)$$

such that

$$L = \Delta((L_\alpha^{(k)})_{\alpha\in\mathcal{P}_k(T_D)}) = \sum_{\alpha\in\mathcal{P}_k(T_D)} L_\alpha^{(k)} \otimes \mathbf{id}_{[\alpha]}$$

holds. From (2.5), each $(L, \mathbf{u}) \in \mathbf{E}(\mathbf{v}) \times \mathcal{O}(\mathbf{v})$ satisfies that

$$(\xi_{\mathbf{v}}^{(k)})^{-1}((L, \mathbf{u})) = \exp(L)(\mathbf{u}) \in \mathcal{U}^{(k)}(\mathbf{v})$$

for $1 \le k \le \operatorname{depth}(T_D)$. Hence the image of $(L, \mathbf{u})$ by $(\xi_{\mathbf{v}}^{(k)})^{-1}$ is independent on the index k. Thus $(\xi_{\mathbf{v}}^{(k)})^{-1}$ is a bijection that maps $\mathbf{E}(\mathbf{v}) \times \mathcal{O}(\mathbf{v})$ onto a subset $\mathcal{W}(\mathbf{v}) \subset \bigcap_{l=1}^{\operatorname{depth}(T_D)} \mathcal{U}^{(l)}(\mathbf{v})$ containing $\mathbf{v}$ for each $1 \le k \le \operatorname{depth}(T_D)$. It allows to we define the bijection

$$\boldsymbol{\xi}_{\mathbf{v}} : \mathcal{W}(\mathbf{v}) \longrightarrow \mathbf{E}(\mathbf{v}) \times \mathcal{O}(\mathbf{v})$$

by $\boldsymbol{\xi}_{\mathbf{v}}(\mathbf{w}) = \xi_{\mathbf{v}}^{(k)}(\exp(L)(\mathbf{u})) = (L, \mathbf{u})$.

Then the following result is straightforward.

Theorem 4.1 *Let T_D be a dimension partition tree with* $\operatorname{depth}(T_D) \ge 2$, *and* $\mathfrak{r} = (r_\alpha)_{\alpha \in T_D} \in \mathcal{AD}(\mathbf{V}_D, T_D)$ *such that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is a proper set of tree-based tensors with a fixed tree-based rank $\mathfrak{r}$. Assume that $(\mathbf{V}_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in T_D \setminus \{D\}$ and that $\|\cdot\|_D$ is a norm on the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{V}_\alpha$ is such that* (2.3) *holds for* $1 \le k \le \operatorname{depth}(T_D)$. *Then the collection*

$$\mathcal{B} = \{(\mathcal{W}(\mathbf{v}), \boldsymbol{\xi}_{\mathbf{v}})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)}$$

is a $\mathcal{C}^\infty$-atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$, and hence $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is a $\mathcal{C}^\infty$-Banach manifold modelled on

$$\mathbf{E} \times \mathbb{R}^{\times_{\alpha \in \mathcal{P}_1(T_D)} r_\alpha},$$

where

$$\mathbf{E} = \bigcap_{k=1}^{\operatorname{depth}(T_D)} \left(\bigoplus_{\alpha \in \mathcal{P}_k(T_D)} \mathcal{L}(\mathbf{U}_\alpha, \mathbf{W}_\alpha) \otimes_a \operatorname{span}\{\mathbf{id}_{[\alpha]}\} \right).$$

Here $\mathbf{U}_\alpha = U_\alpha^{\min}(\mathbf{v})$ is a r_α-dimensional subspace of $\mathbf{V}_\alpha$ for each $\alpha \in T_D \setminus \{D\}$ where $\mathbf{v} \in {}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha$ for $1 \le k \le \operatorname{depth}(T_D)$ *and $\mathbf{W}_\alpha$ is a closed subspace of $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ such that $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = \mathbf{U}_\alpha \oplus \mathbf{W}_\alpha$, where $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is the completion of $\mathbf{V}_\alpha$ for $\alpha \in T_D \setminus \{D\}$.*

4.1 $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ as embedded sub-manifold of $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ for $1 \le k \le \operatorname{depth}(T_D)$

By construction, the natural ambient space of the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is any manifold $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ for $1 \le k \le \operatorname{depth}(T_D)$. We claim that the natural inclusion map $\mathfrak{i} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) \longrightarrow \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ given by $\mathfrak{i}(\mathbf{v}) = \mathbf{v}$ is also written in local coordinates as the natural inclusion map. Indeed, for $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$, the open set $\mathcal{W}(\mathbf{v}) \subset \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ and hence

$$\mathfrak{i} : \mathcal{W}(\mathbf{v}) \longrightarrow \mathcal{U}^{(k)}(\mathbf{v})$$

is the identity map on $\mathcal{W}(\mathbf{v})$, that is, $\mathfrak{i}|_{\mathcal{W}(\mathbf{v})} = id_{\mathcal{W}(\mathbf{v})}$. Thus

$$(\xi_{\mathbf{v}}^{(k)} \circ \mathfrak{i} \circ \boldsymbol{\xi}_{\mathbf{v}}^{-1}) : \mathbf{E}(\mathbf{v}) \times \mathcal{O}(\mathbf{v}) \longrightarrow \mathbf{E}_k(\mathbf{v}) \times \mathfrak{M}_{\mathfrak{r}_k}\left({}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha \right)$$

is the natural inclusion map. Hence its derivative

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} = (\xi_{\mathbf{v}} \circ \mathfrak{i} \circ \boldsymbol{\xi}_{\mathbf{v}}^{-1})'(\boldsymbol{\xi}_{\mathbf{v}}(\mathbf{v})) : \mathbf{E}(\mathbf{v}) \times \left({}_a\bigotimes_{\alpha \in \mathcal{P}_1(T_D)} \mathbf{U}_\alpha \right) \longrightarrow \mathbf{E}_k(\mathbf{v}) \times \left({}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha \right)$$

is also the natural inclusion map which is clearly injective.

In order to prove that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is an embedded sub-manifold of $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ for $1 \le k \le \operatorname{depth}(T_D)$, we need to check the following two conditions for each $1 \le k \le \operatorname{depth}(T_D)$:

(C1) The map $\mathfrak{i}$ should be an immersion. From Proposition 4.1 in [9], it is true when the linear map

$$T_{\mathbf{v}}\mathfrak{i} = (\xi_{\mathbf{v}} \circ \mathfrak{i} \circ \boldsymbol{\xi}_{\mathbf{v}}^{-1})'(\boldsymbol{\xi}_{\mathbf{v}}(\mathbf{v})) : \mathrm{T}_{\mathbf{v}}\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) \longrightarrow \mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$$

is injective and $T_{\mathbf{v}}\mathfrak{i}(\mathrm{T}_{\mathbf{v}}\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)) \in \mathbb{G}\left(\mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)\right)$

(C2) The map

$$\mathfrak{i} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) \longrightarrow \mathfrak{i}\left(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)\right)$$

is a topological homeomorphism.

In consequence, to obtain the desired result we only need to prove that for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ the tangent space

$$\mathrm{T}_{\mathbf{v}}\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D) = \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D) = \mathbf{E}(\mathbf{v}) \times {}_a\bigotimes_{\alpha \in \mathcal{P}_1(T_D)} \mathbf{U}_\alpha$$

belongs to

$$\mathbb{G}\left(\mathbf{E}_k(\mathbf{v}) \times {}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right) = \mathbb{G}\left(\mathbf{E}_k(\mathbf{v})\right) \times \mathbb{G}\left({}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right).$$

Clearly

$${}_a\bigotimes_{\alpha \in \mathcal{P}_1(T_D)} \mathbf{U}_\alpha \in \mathbb{G}\left({}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right),$$

because ${}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha$ is a finite dimensional vector space. From Lemma 2.4 we have

$$\mathbf{E}_k(\mathbf{v}) \in \mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{D_{\|\cdot\|_D}})\right)$$

for $1 \le k \le \mathrm{depth}(T_D)$. The second statement of Lemma 4.14 in [9] implies

$$\mathbf{E}(\mathbf{v}) = \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathbf{E}_k(\mathbf{v}) \in \mathbb{G}\left(\mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{D_{\|\cdot\|_D}})\right).$$

Thus we have the following theorem.

Theorem 4.2 *Let T_D be a dimension partition tree over D and $\mathfrak{r} \in \mathcal{AD}(\mathbf{V}_D, T_D)$ such that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is a proper set of tree-based tensors with a fixed tree-based rank $\mathfrak{r}$. Assume that $(\mathbf{V}_\alpha, \|\cdot\|_\alpha)$ is a normed space for each $\alpha \in T_D \setminus \{D\}$ and let $\|\cdot\|_D$ be a norm on the tensor space $\mathbf{V}_D$ such that (2.3) holds for $1 \le k \le \mathrm{depth}(T_D)$. Then $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is an embedded sub-manifold of $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ for $1 \le k \le \mathrm{depth}(T_D)$.*

Observe that we can also consider the natural inclusion map $\mathfrak{i}$ from $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ to $\mathbf{V}_{D_{\|\cdot\|_D}}$. Under the assumptions of Theorem 4.2, by using Theorem 4.14 of [9], we have that $\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$ is an immersed sub-manifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$ and, for each $\mathbf{v} \in \mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)$, the tangent space

$$\mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D) = \mathbf{E}_k(\mathbf{v}) \times \left({}_a\bigotimes_{\alpha \in \mathcal{P}_k(T_D)} \mathbf{U}_\alpha\right)$$

is linearly isomorphic to the linear space $\mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)\right) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$. Then, by using that $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective, we obtain

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\mathrm{T}_{\mathbf{v}}\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)\right) = \mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)\right) = \bigcap_{k=1}^{\mathrm{depth}(T_D)} \mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\mathrm{T}_{\mathbf{v}}\mathfrak{M}_{\mathfrak{r}_k}(\mathbf{V}_D, T_D)\right) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}}),$$

also by Lemma 4.14 in [9], and it is linearly isomorphic to $\mathrm{T}_{\mathbf{v}}\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$. Thus, also $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D, T_D)$ is an immersed sub-manifold in $\mathbf{V}_{D_{\|\cdot\|_D}}$.

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