**ETICS research school**
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# High-Dimensional Approximation

## Part 2: High-dimensional approximation tools

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# Approximation tools for high-dimensional approximation

We here present classical approximation tools (model classes) for the approximation of multivariate functions

$$u(x_1, \dots, x_d)$$

# Outline

1. Overview of classical approximation tools

2. Approximation theory of (deep) neural networks

3. Approximation theory of tree tensor networks

# Expansions on tensor product bases

- Polynomial models

$$\sum_{\alpha \in \Lambda} a_\alpha x^\alpha, \quad x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$$

where $\Lambda \subset \mathbb{N}^d$ is a set of multi-indices, either fixed (linear approximation) or free (nonlinear approximation).

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- More general expansions on tensorized bases
$$\sum_{\alpha \in \Lambda} a_\alpha \psi_\alpha(x), \quad \psi_\alpha(x) = \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_d}(x_d),$$
e.g. tensor product splines, wavelets...

# Additive and multiplicative models

- Additive models

$$u_1(x_1) + \ldots + u_d(x_d)$$

or more generally

$$\sum_{\alpha \subset \Lambda} u_\alpha(x_\alpha)$$

where $\Lambda \subset 2^{\{1,\ldots,d\}}$ is either fixed (linear approximation) or a free parameter (nonlinear approximation).

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# Separation of variables and tensor networks

- Sum of multiplicative models (canonical tensor format)

$$\sum_{k=1}^{r} v^{(1)}(x_1, k) \dots v^{(d)}(x_d, k)$$

that is a $r$-term approximation from the dictionary of separated functions.

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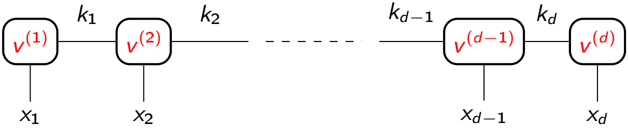
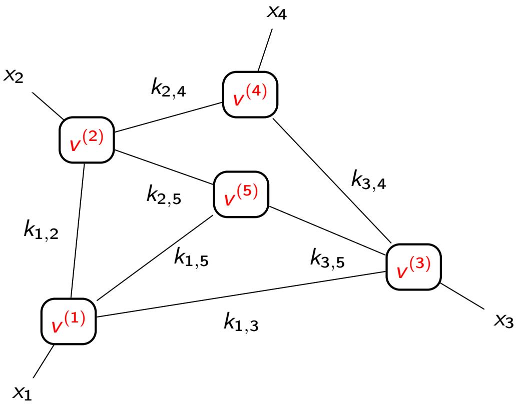
- Tensor train (Matrix Product State)

$$v(x) = \sum_{k_1=1}^{r_1} \dots \sum_{k_{d-1}=1}^{r_{d-1}} v^{(1)}(x_1, k_1) v^{(2)}(k_1, x_2, k_2) \dots v^{(d)}(k_{d-1}, x_d).$$




It is a particular case of tensor networks.

- Tensor networks associated with general graphs



# Composition of functions

$$f(g(x))$$

with $g : \mathbb{R}^d \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$.

$g$ can be seen as a map that extracts $m$ features $g(x)$ (new variables) from an input $x$, that can be fixed (application-dependent) or free.

For linear maps $g(x) = Ax$, this corresponds to ridge approximation

$$f(Ax), \quad A \in \mathbb{R}^{m\times d}$$

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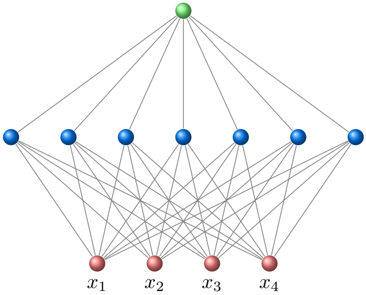
Different regimes

- small $m$, $g$ performs a dimension reduction and $f$ is a low-dimensional function.
- large $m$, $g$ extracts many features and $f$ is expected to be simple, e.g. linear or additive.

A shallow neural network (with one hidden layer of width $m$) is a ridge function

$$a^T \sigma(Ax + b) = \sum_{i=1}^{m} a_i \sigma(\sum_{j=1}^{d} A_{ij} x_j + b_i)$$

where $\sigma$ is a given function (activation function).



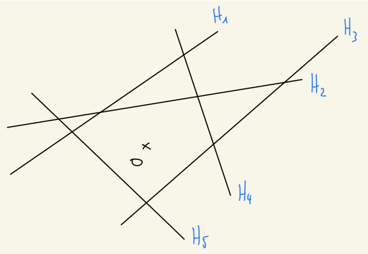
# Neural networks

Classical piecewise polynomial activation functions

- ReLU function $\sigma(t) = \langle t \rangle_+ = \max\{0, t\}$
- RePU(p) function $\sigma(t) = \langle t \rangle_+^p = \max\{0, t\}^p$

ReLU and RePU networks produce a piecewise polynomial approximation (spline) on a free partition of $\mathbb{R}^d$ determined by $m$ hyperplanes

$$H_i = \{x : {w_i}^T x + b_i = 0\}, \quad w_i = (A_{ij})_{j=1}^d \in \mathbb{R}^d$$



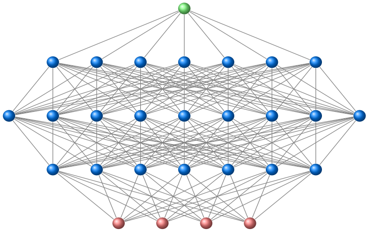
# Deep neural networks

$$T_L \circ \sigma \circ T_{L-1} \circ \ldots \circ T_1 \circ \sigma \circ T_0(x)$$

with $T_\ell : \mathbb{R}^{m_\ell} \to \mathbb{R}^{m_{\ell+1}}$ an affine linear map

$$T_\ell(x) = A_\ell x + b_\ell$$

and $(m_1, \ldots, m_L) \in \mathbb{N}^L$ with $m_0 = d$, $m_{L+1} = 1$.



For ReLU or RePU(p) activation function $\sigma$, the approximation is a piecewise polynomial on a free partition with a number of domains growing exponentially with depth $L$.

## Outline

Different approximation tools $(X_n)_{n\geq 1}$ can be defined depending on which parameters are free (possible architectures) and how complexity is measured.

Letting $\Phi_{L,m}$ be the class of neural networks with depth $L$ and widths $m = (m_1, \dots, m_L)$, we define

$$X_n = \{v \in \Phi_{L,m} : L \in \mathcal{L}, m \in \mathcal{M}_L, compl(v) \leq n\}$$

where *compl* is a complexity measure, $\mathcal{L} \subset \mathbb{N}$ is the set of possible depths and $\mathcal{M}_L \subset \mathbb{N}^L$ the set of possible widths.

Two typical classes of architectures

- Fixed depth $L$ and free width:

$$\mathcal{L} = \{L\}, \quad \mathcal{M}_L = \{(W, \dots, W) : W \in \mathbb{N}\}$$

- Free depth and fixed width $W$:

$$\mathcal{L} = \mathbb{N}, \quad \mathcal{M}_L = \{(W, \dots, W)\}$$

# Approximation tools based on neural networks

For a function $v$ in the class $\Phi_{L,m}$ of neural networks with depth $L$ and widths $m = (m_1, \dots, m_L)$, different measures of complexity:

- number of parameters (fully connected networks)

$$compl_F(v) = \sum_{\ell=0}^{L} m_\ell m_{\ell+1} + m_{\ell+1} \sim W^2 L \text{ for } m_\ell \sim W$$

- number of non-zero parameters (sparsely connected networks)

$$compl_S(v) = \sum_{\ell=0}^{L} \|A_\ell\|_0 + \|b_\ell\|_0$$

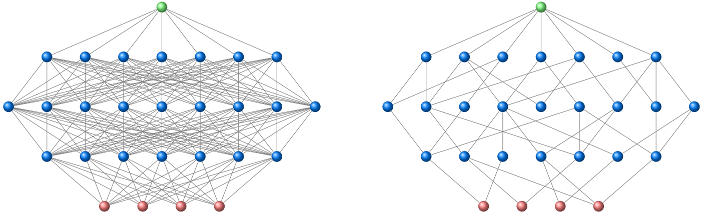


Figure: Fully connected network (left) and Sparsely connected network (right).

Structured sparsity can be imposed (convolutional NN, recurrent NN...) or sparsity pattern can be considered as a free parameter (a challenge on the algorithmic side).

# Deep neural networks approximation theory

Many recent results on the expressivity of deep neural networks for various model classes.

- Approximation classes of deep neural networks (free depth and fixed width) are larger than those of shallow networks (fixed depth and free width) [DeVore et al 2020].

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- Approximation classes of deep ReLU networks are not embedded in standard smoothness classes [Gribonval et al 2021]
- They approximate efficiently functions beyond smoothness classes (discontinuous functions, fractals, refinable functions...)

A few surprises

- For functions $u$ in the unit ball $K$ of $W^{s,\infty}((0,1)^d)$, ReLU networks with free depth can achieve
$$e_n(u)_{L^\infty} \lesssim n^{-p} \quad \text{for arbitrary } p \leq 2s/d.$$
However, since the manifold width $\delta_n(K)_{L^\infty} \gtrsim n^{-s/d}$, a rate $p > s/d$ can be achieved only with discontinuous parameter selection. Also, it requires an encoding of parameters with more than $O(\log_2(\epsilon^{-1}))$ bits to achieve accuracy $\epsilon$.

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Open problems

- Characterize the functions that can be approximated stably with deep networks.
- Characterize functions that can be estimated with partial information and near optimal performance.
- Provide algorithms that achieve near to optimal performance.

## Outline

# Tensor networks

Tensor networks are prominent tools for the representation or approximation of multivariate functions or multidimensional arrays.

- A long history in quantum physics. A zoo of tools exploiting separation of variables (MPS, PEPS, MERA...)
- Tree tensor networks (Hierarchical Tucker tensors) appeared independently in numerical analysis and numerical linear algebra, as an extension of low-rank decompositions to high-order tensors [Hackbusch and Kuhn, Grasedyck, Oseledets and Tyrtyshnikov].
- Growing use in statistics, data science and probabilistic modelling.

# Tensor product of functions

Let $V_\nu \subset \mathbb{R}^{\mathcal{X}_\nu}$ be a space of functions defined on $\mathcal{X}_\nu$.

$\mathcal{X}_\nu$ can be (a subset of) $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, or a set of vectors, sequences, graphs, images...

The tensor product of functions $v^{(\nu)} \in V_\nu$, denoted

$$v = v^{(1)} \otimes \ldots \otimes v^{(d)},$$

is a multivariate function defined on $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d$ and such that

$$v(x_1, \ldots, x_d) = v^{(1)}(x_1) \ldots v^{(d)}(x_d)$$

# Tensor product of functions

The algebraic tensor product of spaces $V_\nu$ is defined as

$$V_1 \otimes \ldots \otimes V_d = \mathrm{span}\{v^{(1)} \otimes \ldots \otimes v^{(d)} : v^{(\nu)} \in V_\nu, 1 \leq \nu \leq d\}$$

which is the space of multivariate functions $v$ which can be written as a finite linear combination of elementary (separated functions), i.e.

$$v(x_1, \ldots, x_d) = \sum_{k=1}^{r} v_k^{(1)}(x_1) \ldots v_k^{(d)}(x_d).$$

# Rank of multivariate functions and canonical format

The canonical rank of a multivariate function $f(x_1, \ldots, x_d)$ is the minimal integer such that $f$ has a representation

$$f(x) = \sum_{k=1}^{r} v_k^1(x_1) \ldots v_k^d(x_d)$$

Given a finite-dimensional tensor space $V = V_1 \otimes \ldots \otimes V_d$ of multivariate functions we define a canonical tensor format in $V$ as a set of functions

$$\mathcal{R}_r(V) = \{f \in V : \mathrm{rank}(f) \leq r\}$$

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From the practical point of view, it is not a nice format. In particular, $\mathcal{R}_r(V)$ is not closed for $d \geq 3$ and $r \geq 2$.

For any continuous parametrization $\mathcal{R}_r(V) = \{v = R(p) : p \in P\}$, and for any tensor of $v \in \overline{\mathcal{R}_r(V)} \setminus \mathcal{R}_r(V)$ of border rank $r$, the quantity

$$\delta(v, \epsilon) = \inf\{\|p\| : \|v - R(p)\| < \epsilon\}$$

diverges as $\epsilon \to 0$ [Hackbusch 2021].

A multivariate function $f(x_1, \ldots, x_d)$, for any set $\alpha \subset \{1, \ldots, d\}$, can be identified with a bivariate function $f(x_\alpha, x_{\alpha^c})$ of two complementary subsets of variables.

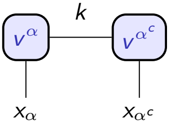
The rank of the bivariate function $f(x_\alpha, x_{\alpha^c})$ is the $\alpha$-rank of $f$, denoted $\mathrm{rank}_\alpha(f)$.

A function with $\alpha$-rank bounded by $r_\alpha$ admits a representation

$$f(x) = \sum_{k=1}^{r_\alpha} v_k^\alpha(x_\alpha) v_k^{\alpha^c}(x_{\alpha^c})$$

or using tensor diagram notations

$f(x) =$




where a connection between two tensors represents a contraction along one mode of each tensor.

# $\alpha$-rank

## Example

- $u(x) = u^1(x_1)\dots u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha)u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \prod_{\nu\in\alpha} u^\nu(x_\nu)$. Therefore, for any $\alpha$, $\text{rank}_\alpha(u) = 1$.

- $u(x) = \sum_{k=1}^r u_k^1(x_1)\dots u_k^d(x_d)$ can be written $\sum_{k=1}^r u_k^\alpha(x_\alpha)u_k^{\alpha^c}(x_{\alpha^c})$ with $u_k^\alpha(x_\alpha) = \prod_{\nu\in\alpha} u_k^\nu(x_\nu)$. Therefore, for any $\alpha$, $\text{rank}_\alpha(u) \le r$, with equality if the functions $\{u_k^\alpha(x_\alpha)\}$ and the functions $\{u_k^{\alpha^c}(x_{\alpha^c})\}$ are linearity independent.

  We deduce the following relation between $\alpha$-ranks and canonical rank:

  $$\text{rank}_\alpha(u) \le \text{rank}(u), \quad \text{for any } \alpha.$$

- $u(x) = u^1(x_1) + \dots + u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha) + u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \sum_{\nu\in\alpha} u^\nu(x_\nu)$. Therefore, $\text{rank}_\alpha(u) \le 2$.

## Low-rank tensor format

Given

- a finite-dimensional tensor space $V = V_1 \otimes \ldots \otimes V_d$ of multivariate functions
- a collection $T$ of subsets in $\{1, \ldots, d\}$,
- a tuple of ranks $r = (r_\alpha)_{\alpha \in T}$,

we define a low-rank tensor format in $V$ as a set of functions

$$\mathcal{T}_r^T(V) = \{f \in V : \mathrm{rank}_\alpha(f) \leq r_\alpha, \alpha \in T\}$$

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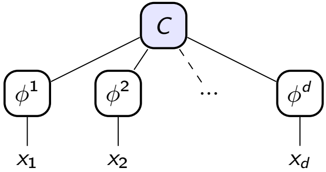
with representation

$$f(x) = \sum_{i_1 \in I_1} \ldots \sum_{i_d \in I_d} C(i_1, \ldots, i_d)\phi^1(x_1)_{i_1} \ldots \phi^d(x_d)_{i_d} =$$




where $\phi^\nu$ is a feature map associated with $V^\nu$ and $C \in \mathbb{R}^{I_1 \times \ldots \times I_d}$ is a rank-structured algebraic tensor.

# Tensor train format

With

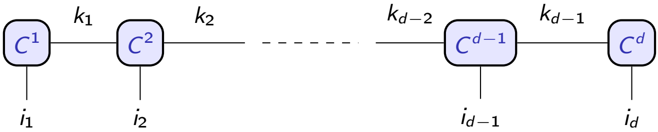
$$T = \{\{1\}, \{1,2\}, ..., \{1,\ldots,d\}\},$$

$\mathcal{T}_r^T(V)$ coincides with the tensor train format.

A function $f$ in $\mathcal{T}_r^T(V)$ has coefficients

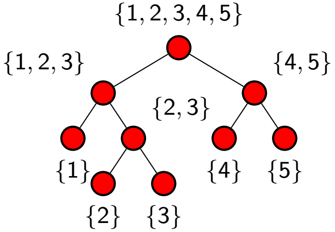
$$C(i_1,\ldots,i_d) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{d-1}=1}^{r_{d-1}} C^1(i_1,k_1)C^2(k_1,i_2,k_2)\ldots C^d(k_{d-1},i_d).$$

$=$

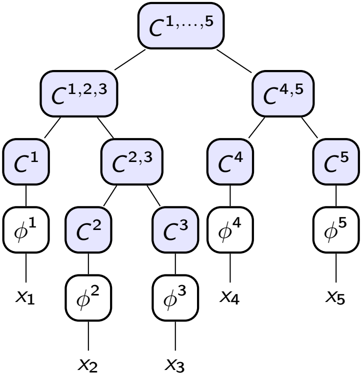


# Hierarchical Tucker format (Tree tensor networks)

If $T$ is a dimension partition tree, $\mathcal{T}_r^T(V)$ is a tree-based (or hierarchical) tensor format and a function in $\mathcal{T}_r^T(V)$ admits a multilinear parametrization with a collection of parameters $\{C^\alpha : \alpha \in T\}$ forming a tree tensor network.




Dimension tree $T$




Tree tensor network

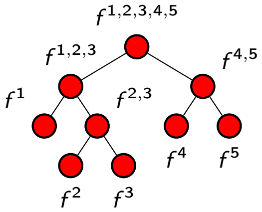
# Tree tensor networks as a compositional function network

By identifying a tensor $C^{(\alpha)} \in \mathbb{R}^{n_1 \times \ldots \times n_s \times r_\alpha}$ with a $\mathbb{R}^{r_\alpha}$-valued multilinear function

$$f^{(\alpha)} : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s} \to \mathbb{R}^{r_\alpha},$$

a function $v$ in $\mathcal{T}_r^T(V)$ admits a representation as a tree-structured composition of multilinear functions $\{f^{(\alpha)}\}_{\alpha \in T}$, e.g.

$$v(x) = f^D(f^{1,2,3}(f^1(\phi^1(x_1)), f^{2,3}(f^2(\phi^2(x_2)), f^3(\phi^3(x_3))), f^{4,5}(f^4(\phi^4(x_4)), f^5(\phi^5(x_5))))$$



A multilinear map $f^\alpha$ can also be written

$$f^\alpha(z_1, \dots, z_s) = A^\alpha \sigma(z_1, \dots, z_d), \quad z_k \in \mathbb{R}^{n_k},$$

with a matrix

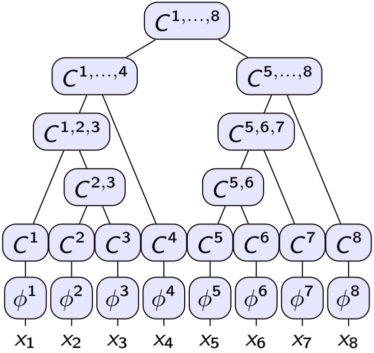
$$A^\alpha \in \mathbb{R}^{r_\alpha \times N}, \quad N = n_1 \dots n_s$$

and a fixed multilinear function

$$\sigma(z_1, \dots, z_s) = vec(z_1 \otimes \dots \otimes z_s) \in \mathbb{R}^N$$

# Tree tensor networks as feed-forward neural networks

It corresponds to a sum-product feed forward neural network with a sparse architecture (given by $T$), a number of hidden layers equal to $depth(T)+1$ (including a featuring layer), and width at level $\ell$ related to the $\alpha$-ranks of the nodes $\alpha$ of level $\ell$.



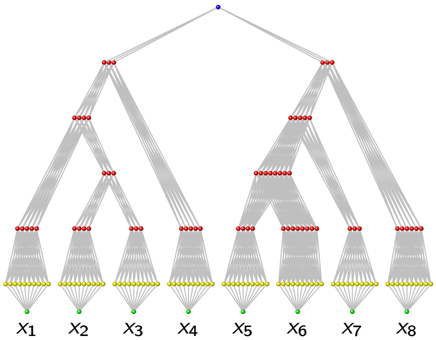




Figure: Tree tensor network and corresponding feed-forward sum-product neural network with 10 features per variable $x_\nu$ (right)

# Outline

# Approximation tools based on tree tensor networks

For the approximation of a function, a first approach is to introduce subspaces $V^\nu_{N_\nu}$ of finite dimension (e.g. polynomials, splines, wavelets, RKHS...) and consider tree tensor networks $f \in \mathcal{T}_r^T(V_N)$ where

$$V_N = V^1_{N_1} \otimes \ldots \otimes V^d_{N_d},$$

with variable $N$ and $r$.

Spaces $V^\nu_{N_\nu}$ have to be well chosen, e.g. polynomials for analytic functions, splines with a degree adapted to the regularity of the function...

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \mathcal{T}_r^T(V_N) : N \in \mathbb{N}^d, r \in \mathbb{N}^T, \mathit{compl}(f) \leq n\}.$$

The dimensions $N$ and the ranks $r$ are free parameters, and $\mathit{compl}(\cdot)$ is some complexity measure.

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

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The dimensions $N$ and the ranks $r$ are free parameters, and $\mathit{compl}(\cdot)$ is some complexity measure.

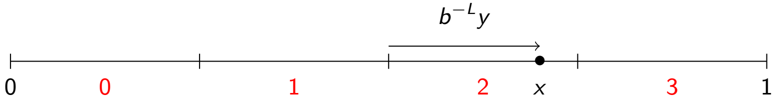
An alternative approach is to rely on tensorization of functions (specific featuring step).

# Tensorization of univariate functions

Consider a function $f \in \mathbb{R}^{[0,1)}$ defined on the interval $[0,1)$.

- For $b, L \in \mathbb{N}$, we subdivide uniformly the interval $[0,1)$ into $b^L$ intervals. Any $x \in [0,1)$ can be written

$$x = b^{-L}(i + y), \quad i \in \{0, \ldots, b^L - 1\}, \quad y \in [0,1).$$



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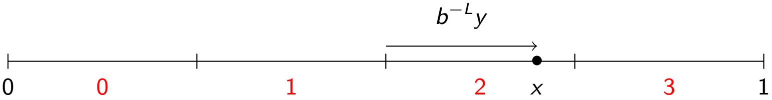
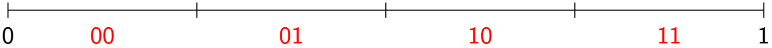



- The integer $i$ admits a representation in base $b$

$$i = \sum_{k=1}^{L} i_k b^{L-k} = [i_1 \dots i_L]_b, \quad i_k \in \{0, \dots, b-1\}$$



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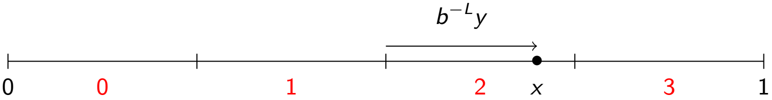


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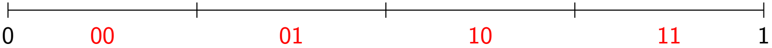
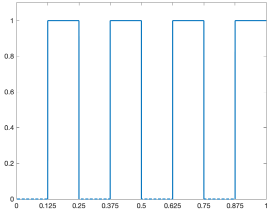



- $f$ is thus identified with a multivariate function (tensor of order $L+1$)

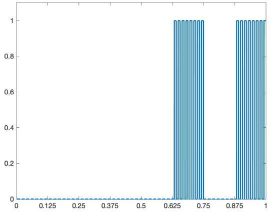
$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes L} \otimes \mathbb{R}^{[0,1)} \quad \text{such that} \quad f(x) = \boldsymbol{f}(i_1, \dots, i_L, y)$$

# Tensorization of univariate functions

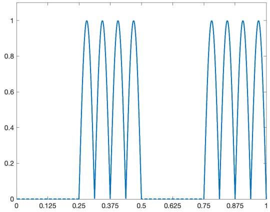
Digit $i_k(x)$ can be seen as a particular feature extracted from $x$.



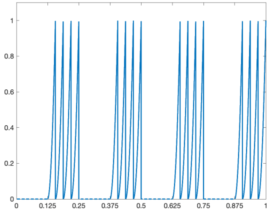

(a) $i_{\mathbf{3}}(x)$




(b) $i_{\mathbf{1}}(x)i_{\mathbf{3}}(x)i_{\mathbf{7}}(x)$




(c) $i_{\mathbf{2}}(x)sin(\pi y)$ $(L = 4)$




(d) $i_{\mathbf{3}}(x)y^{\mathbf{2}}$ $(L = 5)$

# Tensorization of multivariate functions

A function $f(x_1, \dots, x_d)$ defined on $[0,1)^d$ can be similarly identified with a tensor of order $(L+1)d$

$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes Ld} \otimes (\mathbb{R}^{[0,1)})^{\otimes d}$$

such that

$$f(x_1, \dots, x_d) = \boldsymbol{f}(i_1^1, \dots, i_1^L, y_1, \dots, i_d^1, \dots, i_d^L, y_d)$$

$$\text{where} \quad x_\nu = b^{-L}(\sum_{k=1}^{L} i_\nu^k b^{L-k} + y_\nu) = [0.i_\nu^1 \dots i_\nu^L]_b + b^{-L} y_\nu$$

Digits $(i_1^1, \dots, i_1^L, \dots, i_d^1, \dots, i_d^L)$ encode a uniform partition of $[0,1)^d$ into $b^{dL}$ elements.

| | | | |
|---|---|---|---|
| 0011 | 0111 | 1011 | 1111 |
| 0010 | 0110 | 1010 | 1110 |
| 0001 | 0101 | 1001 | 1101 |
| 0000 | 0100 | 1000 | 1100 |

Figure: d=2, b=2, L=2

# Tensorization of multivariate functions

Using a different (resolution-wise) ordering of variables, the function can be identified with another tensor

$$f(x_1, \dots, x_d) = \boldsymbol{f}(i_1^1, \dots, i_d^1, \dots, i_1^L, \dots, i_d^L, y_1, \dots, y_d)$$

It corresponds to another encoding of the partition of $[0, 1)^d$ into $b^{dL}$ elements.

| | | | |
|---|---|---|---|
| 0101 | 0111 | 1101 | 1111 |
| 0100 | 0110 | 1100 | 1110 |
| 0001 | 0011 | 1001 | 1011 |
| 0000 | 0010 | 1000 | 1010 |

Figure: d=2, b=2, L=2

# Tensorization of multivariate functions

This particular re-parametrization is related to Morton space filling curve (or Z-order), which consists in mapping a point

$$([0.i_1^1 \dots i_1^L \dots]_2, \dots, [0.i_d^1 \dots i_d^L \dots]_2) \in [0,1]^d$$

to a real number

$$[0.i_1^1 \dots i_d^1 \dots i_1^L \dots i_d^L \dots]_2 \in [0,1]$$

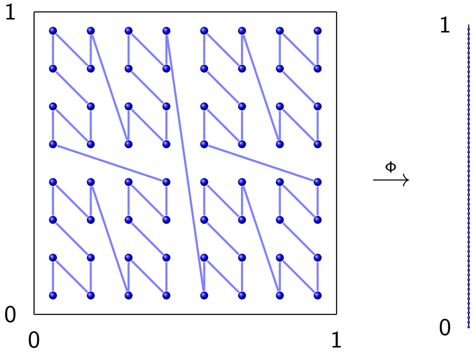
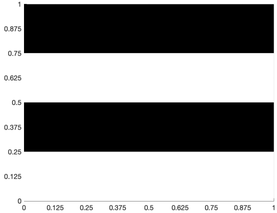


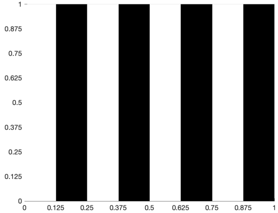

Figure: $b = 2$ and $L = 3$

# Tensorization of multivariate functions

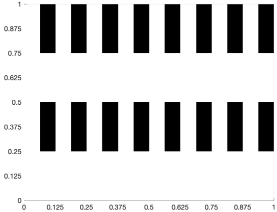
Digit $i_\nu^\ell(x)$ can be seen as a particular feature extracted from $x$.



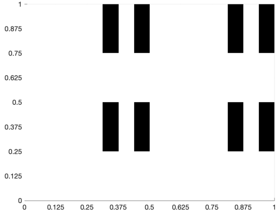

(a) $i_{\mathbf{1}}^{\mathbf{2}}(x)$




(b) $i_{\mathbf{2}}^{\mathbf{1}}(x)$




(c) $i_{\mathbf{1}}^{\mathbf{2}}(x) i_{\mathbf{3}}^{\mathbf{1}}(x)$




(d) $i_{\mathbf{1}}^{\mathbf{1}}(x) i_{\mathbf{1}}^{\mathbf{2}}(x) i_{\mathbf{3}}^{\mathbf{1}}(x)$

# Tensorization of multivariate functions

The map $T_{b,L}$ which associates to a function $f$ its tensorization $\boldsymbol{f}$ is a linear isometry from $L^p([0,1)^d)$ to $L^p(\{0,\dots,b-1\}^{Ld} \times [0,1)^d)$ for any $0 < p \leq \infty$.

We consider functions whose tensorization at resolution $L$ are in the tensor space

$$\boldsymbol{V}_L = (\mathbb{R}^b)^{\otimes Ld} \otimes S^{\otimes d}$$

with $S \subset \mathbb{R}^{[0,1)}$ some subspace of univariate functions, invariant through $b$-adic dilation.

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If $S = \mathbb{P}_m$, $V_L = T_{b,L}^{-1}(\boldsymbol{V}_L)$ is identified with the space of multivariate splines of degree $m$ over a uniform partition with $b^{dL}$ elements, i.e.

$$V_L = V_{N_1}^1 \otimes \ldots \otimes V_{N_d}^d$$

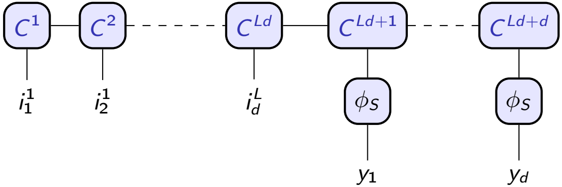
with $N_1 = ... = N_d = b^L$ and $V_{N_\nu}^\nu$ a space of univariate splines of degree $m$ over a uniform partition with $N_\nu = b^L$ intervals.

# Approximation tools based on tree tensor networks

Then as an approximation tool, we consider functions $f$ whose tensorization is a tensor network in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$, with $T_L$ a dimension tree over $\{1, \dots, Ld + d\}$.

Using the tensor train format, the corresponding function $f(x_1, \dots, x_d)$ has the representation

$f(x_1, \dots, x_d) =$




with $\phi_S$ the feature map associated with $S$. This is closely related to the quantized tensor train (QTT) format [Kazeev, Khoromskij, Oseledets, Schwab, ...]

Later on, we consider $S = \mathbb{P}_m$ and $\phi_S(y) = (1, y, ..., y^{m+1})$ or any other polynomial basis.

# Approximation tools based on tree tensor networks

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \Phi_{L,T_L,r} : L \in \mathbb{N}_0, r \in \mathbb{N}^{T_L}, compl(f) \le n\}$$

with $\Phi_{L,T_L,r}$ the functions whose tensorization at resolution $L$ is in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$.

The resolution $L$ and ranks $r$ are free parameters, and $compl(\cdot)$ is some complexity measure.

The complexity $compl(f)$ of $f$ is defined as the complexity of the associated tensor network $\{C^\alpha\}_{\alpha\in T}$.

- Number of parameters (full tensor network)

$$compl_{\mathcal{F}}(f) = \sum_\alpha \text{number\_of\_entries}(C^\alpha)$$

- Number of non-zero parameters (sparse tensor network)

$$compl_{\mathcal{S}}(f) = \sum_\alpha \|C^\alpha\|_0$$

# Complexity measures and corresponding approximation tools

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- Number of non-zero parameters (sparse tensor network)

$$compl_{\mathcal{S}}(f) = \sum_{\alpha} \|C^\alpha\|_0$$

Complexity measures $compl_{\mathcal{F}}$ and $compl_{\mathcal{S}}$ yield two different approximation tools

$$\Phi_n^{\mathcal{F}} \quad \text{and} \quad \Phi_n^{\mathcal{S}}$$

such that

$$\Phi_n^{\mathcal{F}} \subset \Phi_n^{\mathcal{S}} \subset \Phi_{a+bn^2}^{\mathcal{F}}$$

## Approximation theory of tree tensor networks

Given a function $f$ from a Banach space $X$, the best approximation error of $f$ by an element of $\Phi_n$ is

$$E(f, \Phi_n)_X := \inf_{g \in \Phi_n} \|f - g\|_X$$

Fundamental questions are:

- does $E(f, \Phi_n)_X$ converge to 0 for any $f$ ?
  (universality)
- does a best approximation exist ?
  (proximinality)
- how fast does it converge for functions from classical function classes ?
  (expressivity)
- what are the functions for which $E(f, \Phi_n)_X$ converges with some given rate ?
  (characterization of approximation classes)

# Outline

# Universality

First note that for any algebraic feature tensor space $V$, and any tree $T$,

$$\bigcup_r \mathcal{T}_r^T(V) = V.$$

so the question of universality of tree tensor networks boils down to conditions on the tensor feature spaces.

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so the question of universality of tree tensor networks boils down to conditions on the tensor feature spaces.

- Consider the first family of approximation tools with variable feature spaces $V_N$, $N \in \mathbb{N}^d$.

  If $\bigcup_N V_N$ is dense in $X$, then the tools are universal for functions in $X$.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, and for polynomial or splines spaces $V_N$.

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  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, and for polynomial or splines spaces $V_N$.

- Consider the second family of approximation tools using tensorization.

  If $\bigcup_L V_L$ is dense in $X$, then the tools are universal for functions in $X$.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, assuming that $S$ contains the function one.

For any tree $T$, any $T$-rank $r$, and any finite dimensional tensor space $V$ of $X$, $\mathcal{T}_r^T(V)$ is a closed set in $V$.

$\Phi_n^{\mathcal{F}}$ (full tensor networks) is a finite union of such sets, all contained in a single finite dimensional space $V^*$. Then $\Phi_n^{\mathcal{F}}$ is a closed set of a finite dimensional space $V^*$ and is therefore proximinal in $X$.

However, $\Phi_n^{\mathcal{F}}$ (sparse tensor networks) is not closed.

Different ways to analyse the expressivity of tree tensor networks

- Exploit known results on other approximation tools and estimate the complexity to encode these tools using tree tensor networks.
- Directly encode a function using tree tensor networks (with controlled errors)
- Analyse the convergence of bilinear approximations

$$u(x_\alpha, x_{\alpha^c}) \approx \sum_{k=1}^{r_\alpha} u_k^\alpha(x_\alpha) u_k^{\alpha^c}(x_{\alpha^c})$$

or the approximability of partial evaluations $u(\cdot, x_{\alpha^c})$ by linear approximation spaces of dimension $r_\alpha$.

# Encoding polynomials and splines

## Polynomials

The tensorization of a polynomial of degree $p$ has all ranks bounded by $p+1$.

## Trigonometric polynomials

The tensorization of the function $\cos(\omega x + \varphi)$ has all ranks equal to 2.

Then the tensorization of a trigonometric polynomial of degree $p$ has all ranks bounded by $2p+1$.

## Free knot splines

A spline $\varphi$ of degree $p$ over $N$ $b$-adic intervals forming a partition of $[0,1)$ is such that

$$\mathrm{rank}_{\{1,\dots,\nu\}}(\varphi) \le \begin{cases} p+N, & 1 \le \nu < \ell. \\ p+1, & \ell \le \nu \le L. \end{cases}$$

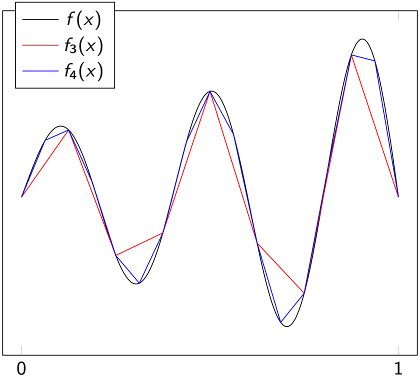
where $b^{-\ell}$ is the minimal length of intervals.

# Encoding polynomials and splines

## Ranks of interpolants

For a function $f$ and its interpolation $f_L$ onto $V_L$, the space of piecewise polynomials of degree $m$ on a uniform partition of $b^L$ intervals, it holds

$$\mathrm{rank}_\alpha(\boldsymbol{f}_L) \leq \mathrm{rank}_\alpha(\boldsymbol{f})$$



# Encoding multi-resolution analysis

For a function $\psi : \mathbb{R} \to \mathbb{R}$ supported on $[0, 1]$, we define its level $\ell$ *b*-adic dilation, shifted by $j = 0, \ldots, b^L - 1$,

$$\psi_{\ell,j}(x) = \psi(b^\ell x - j)$$

Its tensorization at level $\ell$ is an elementary (rank-one) tensor

$$T_{b,\ell}\psi_{\ell,j} = e_{j_1} \otimes \ldots e_{j_\ell} \otimes \psi$$

with $j = [j_1, \ldots, j_\ell]_b$ and $e_k$ the canonical basis vectors in $\mathbb{R}^b$.

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Its tensorization at level $L \geq l$ is

$$T_{b,L}\psi_{\ell,j} = e_{j_1} \otimes \dots e_{j_\ell} \otimes (T_{b,L-\ell}\psi)$$

The (approximate) encoding of $\psi_{\ell,j}$ boils down to the (approximate) encoding of the mother function $\psi$ with tensor networks.

In particular, if $\psi$ is a (piecewise) polynomial, $\psi_{\ell,j}$ is encoded at precision $\epsilon$ using tensorization at level $L = \ell + O(\log(\epsilon^{-1}))$.

This yields a very efficient encoding of piecewise polynomial MRAs (B-spline wavelets).

# Approximation of functions from Besov spaces $B^\alpha_q(L^p)$

From results on spline approximation and their encoding with tensor networks, we obtain

**Theorem**

*Let $f \in B^\alpha_q(L^p)$ with $\alpha > 0$ and $0 < p, q \leq \infty$. Then*

$$E(f, \Phi_n^{\mathcal{F}})_{L^p} \leq C n^{-\alpha/d} |f|_{B^\alpha_\infty(L^p)}$$

- Tensor networks achieve optimal rates for any Besov regularity order (measured in $L^p$ norm).
- They perform as well as optimal linear approximation tools (e.g. splines), without requiring to adapt the tool to the regularity order $\alpha$.
- The depth (resolution $L$) of the network is crucial to capture extra regularity ($\alpha > m + 1$).

# Approximation of functions from Besov spaces $B_q^\alpha(L^\tau)$

Now consider the harder problem of approximating functions from Besov spaces $B_q^\alpha(L^\tau)$ where regularity is measured in a $L^\tau$-norm weaker than $L^p$-norm.

From results on best $n$-term approximation using dilated splines, we obtain

**Theorem**

*Let $f \in B_q^\alpha(L^\tau)$ with $\alpha > 0$, $0 < q \leq \tau < p < \infty$, $1 \leq p < \infty$ and*

$$\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}.$$

*Then*

$$E(f, \Phi_n^{\mathcal{S}})_{L^p} \leq C n^{-\tilde{\alpha}/d} |f|_{B_q^\alpha(L^\tau)}, \quad E(f, \Phi_n^{\mathcal{F}})_{L^p} \leq C n^{-\tilde{\alpha}/(2d)} |f|_{B_q^\alpha(L^\tau)},$$

*for arbitrary $\tilde{\alpha} < \alpha$.*

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*for arbitrary $\tilde{\alpha} < \alpha$.*

- Sparse tensor networks achieve arbitrarily close to optimal rates in $O(n^{-\alpha/d})$ for functions with any Besov smoothness $\alpha$ (measured in $L^\tau$ norm), without the need to adapt the tool to the regularity order $\alpha$.
- Here depth and sparsity are crucial for obtaining near to optimal performance.
- Full tensor networks have slightly lower performance in $O(n^{-\alpha/(2d)})$.

# High-dimensional approximation

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- For Besov spaces $B_q^\alpha(L^p)$, tensor networks achieve (near to) optimal rate in $O(n^{-\alpha/d})$ which deteriorates with $d$, that is the curse of dimensionality.
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- For Besov spaces with anisotropic smoothness $AB_q^{\boldsymbol{\alpha}}(L^p)$, sparse tensor networks also achieve near to optimal rates in $O(n^{-s(\boldsymbol{\alpha})/d})$ with

$$s(\boldsymbol{\alpha})/d = (\alpha_1^{-1} + \ldots + \alpha_d^{-1})^{-1}$$

  the aggregated smoothness. Curse of dimensionality can be circumvented with sufficient anisotropy.

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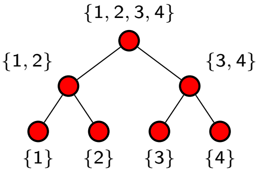
  the aggregated smoothness. Curse of dimensionality can be circumvented with sufficient anisotropy.
- Curse of dimensionality can be circumvented for non usual function classes such as compositions of smooth functions

# Compositional functions

Consider a tree-structured composition of smooth functions $\{f_\alpha : \alpha \in T\}$, see [Mhaskar, Liao, Poggio 2016] for deep neural networks, and [Bachmayr Nouy and Schneider 2021] for tree tensor networks.

$$f_{1,2,3,4}\left(f_{1,2}\left(f_1(x_1), f_2(x_2)\right), f_{3,4}\left(f_3(x_3), f_4(x_4)\right)\right)$$



# Compositional functions

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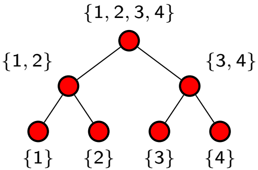
$$f_{1,2,3,4}\left(f_{1,2}\left(f_1(x_1), f_2(x_2)\right), f_{3,4}\left(f_3(x_3), f_4(x_4)\right)\right)$$




Assuming that the functions $f_\alpha \in W^{k,\infty}$ with $\|f_\alpha\|_{L^\infty} \leq 1$ and $\|f_\alpha\|_{W^{1,\infty}} \leq B$, the complexity to achieve an accuracy $\epsilon$

$$n(\epsilon) \lesssim \epsilon^{-3/k}(L+1)^3 B^{3L} d^{1+3/2k}$$

with $L = \log_2(d)$ for a balanced tree and $L + 1 = d$ for a linear tree.

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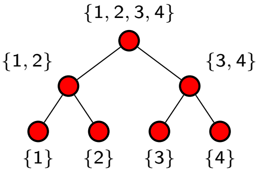


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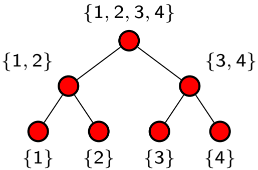


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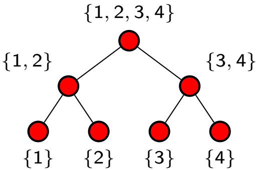


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- For $B \leq 1$, the complexity only scales polynomially in $d$ whatever the tree: no curse of dimensionality !

# More regularity, analytic functions

For function $f : [0,1]$ with analytic extension on an open complex domain

$$D_\rho = \{z \in \mathbb{C} : dist(z, [0,1]) < \frac{\rho - 1}{2}\}, \quad \rho > 1,$$

we obtain an exponential convergence

$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \leq C\gamma^{-n^{\mathbf{1/3}}},$$

with $\gamma = \min\{\rho, b^{(m+1)/b}\}$.

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The proof relies on the approximation of analytic functions with polynomials and the encoding of polynomials with tree tensor networks: a chebychev polynomial $p$ of deree $\bar{m}$ is such that

$$\|f-p\|_{L^\infty} \le \frac{2}{\rho-1}\|f\|_{L^\infty(D_\rho)}\rho^{-\bar{m}}$$

A polynomial of degree $\bar{m}$ can be approximated by $\varphi$ in $\Phi_{L,r,m}$ with an error in $O(b^{-L(m+1)})$, so that

$$\|f-\varphi\|_{L^\infty} \lesssim \rho^{-\bar{m}} + b^{-L(m+1)}$$

We obtain the result by choosing $\bar{m} \sim n^{1/3}$ and $L \sim b^{-1}n^{1/3}$, so that $compl_{\mathcal{F}}(\varphi) \le n$.

# Analytic functions with singularities

Consider the approximation of $u(x) = x^\alpha$, $0 < \alpha \leq 1$, in $L^\infty$.

- Piecewise constant linear approximation.

$$u \in B^\alpha_\infty(L^\infty), \quad u \notin B^\beta_\infty(L^\infty) \quad \text{for } \beta > \alpha,$$

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- Piecewise constant approximation and tensor networks.
  A piecewise constant approximation on a uniform mesh with $2^L$ elements exploiting low-rank structures gives an exponential convergence

$$E(f, \Phi_n^{\mathcal{F}}) \leq C\beta^{-n^\gamma}$$

  Achieves almost the performance of *h-p* methods [Kazeev and Schwab].

# Beyond smoothness

Consider the Weierstrass function, continuous but nowhere differentiable

$$f(x) = \sum_{k=0}^{\infty} a^{-\alpha k} \cos(a^k \pi x), \quad a > 0, \quad 0 < \alpha \leq 1,$$

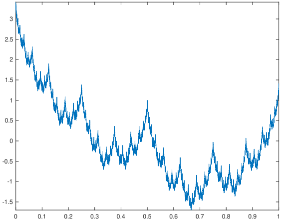



Figure: Weierstrass function for $\alpha = 1/2$ , $a = 2$

We have an exponential convergence in $L^\infty$-norm

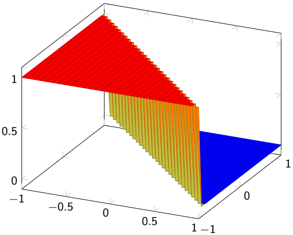
$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \lesssim \beta^{-n^{1/3}}$$

An error $\epsilon$ is achieved with resolution $L \sim \log(\epsilon^{-1})$, ranks $\sim \log(\epsilon^{-1})$ and complexity $n \sim \log(\epsilon^{-1})^3$

# Discontinuous functions: the power of tensorization

Consider the problem of approximating the bivariate function on $(-1,1)^2$

$$u(x,t) = \begin{cases} 1 & \text{if } x+t<0 \\ 0 & \text{if } x+t \geq 0 \end{cases}$$




The manifold $K = \{u(\cdot,t) : t \in (-1,1)\}$ contains the indicator functions $1_{[-1,x_i]}(x)$, $x_i = -1 + 2i/m$. Therefore the balanced convex hull of $K$ contains the orthogonal system $S = \{\psi_i(x) = \frac{1}{2} 1_{(x_i,x_{i+1}]}(x) : 1 \leq i \leq m\}$ with $\|\psi_i\|_{L^2} = (2m)^{-1/2}$ and by taking $m = 2n$, we deduce

$$d_n(K)_{L^2} \geq 1/(2\sqrt{2})n^{-1/2},$$

so that the best rank-$n$ approximation

$$u_n(x,t) = \sum_{i=1}^{n} v_i(x) w_i(t)$$

does not converge better than $\|u - u_n\|_{L^2} \gtrsim n^{-1/2}$.

# Discontinuous functions: the power of tensorization

A piecewise constant interpolant $u^L$ on a uniform grid with mesh size $2^{-L}$ is such that

$$\|u - u^L\|_{L^2} \leq meas(\{(x,t) : u \neq u^L\})^{1/2} \leq 2^{1/2} 2^{-L/2}$$

Using a tensorization $\tilde{\boldsymbol{u}}^L(i_1^x, ..., i_L^x, i_1^t, ..., i_L^t)$, we have

$$rank_{\{1,\ldots,L\}}(\tilde{\boldsymbol{u}}^L) = \operatorname{rank} u_L \sim 2^L$$

that means an encoding complexity in tensor train format $compl(\tilde{\boldsymbol{u}}^L) \gtrsim 2^{2L}$, which yields an approximation error $\gtrsim n^{-1/4}$.

However, the tensorization $\boldsymbol{u}^L(i_1^x, i_1^t, ..., i_L^x, i_L^t)$ of $u^L(x,t)$ satisfies

$$rank_{\{1,\ldots,\nu\}}(\boldsymbol{u}^L) \leq 3$$

for all $\nu$. Therefore, using tensor train format, $compl(u^L) \leq 36L$ and

$$E(u, \Phi_n^{\mathcal{F}})_{L^2} \leq 2^{1/2} 2^{-n/72}.$$

# Outline

# Canonical versus tree-based format

Consider a finite dimensional tensor space $V = V^1 \otimes \ldots \otimes V^d$ with $\dim(V_\nu) = \mathbb{R}^N$, which is identified with $\mathbb{R}^{N\times\cdots\times N}$. Denote by $\mathcal{R}_r = \{v : \text{rank}(v) \leq r\}$ and $\mathcal{T}_r^T = \{v : \text{rank}_\alpha(v) \leq r, \alpha \in T\}$.

- From canonical format to tree-based format.
  For any $v$ in $V$ and any $\alpha \subset D$, the $\alpha$-rank is bounded by the canonical rank:

$$\text{rank}_\alpha(v) \leq \text{rank}(v).$$

  Therefore, for any tree $T$,

$$\mathcal{R}_r \subset \mathcal{T}_r^T,$$

  so that an element in $\mathcal{R}_r$ with storage complexity $O(dNr)$ admits a representation in $\mathcal{T}_r^T$ with a storage complexity $O(dNr + dr^{s+1})$ where $s$ is the arity of the tree $T$.

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- From tree-based format to canonical format. For a balanced or linear binary tree, the subset
  $$S = \{v \in \mathcal{T}_r^T : \mathrm{rank}(v) < q^{d/2}\}, \quad q = \min\{N, r\},$$
  is of Lebesgue measure 0.

  Then a typical element $v \in \mathcal{T}_r^T$ with storage complexity of order $dNr + dr^3$ admits a representation in canonical format with a storage complexity of order $dNq^{d/2}$.

## Influence of the tree

- For some functions, the choice of tree is not crucial. For example, an additive function
$$u_1(x_1) + \ldots + u_d(x_d)$$
has $\alpha$-ranks equal to 2 whatever $\alpha \subset D$.

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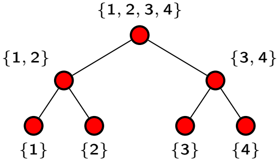
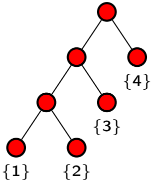
- But usually, different trees lead to different complexities of representations.




$T^B$ (Balanced tree)




$T^L$ (Linear tree)

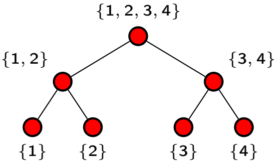
- If $\text{rank}_{T^L}(u) \leq r$ then $\text{rank}_{T^B}(u) \leq r^2$
- If $\text{rank}_{T^B}(u) \leq r$ then $\text{rank}_{T^L}(u) \leq r^{\log_2(d)/2}$

# Influence of the tree

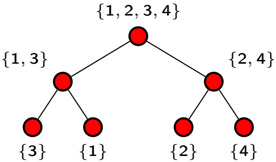
Given a tree $T$ and a permutation $\sigma$ of $D = \{1, \dots, d\}$, we define a tree $T_\sigma$

$$T_\sigma = \{\sigma(\alpha) : \alpha \in T\}$$

having the same structure as $T$ but different nodes.




$T$




$T_\sigma$ with $\sigma = (3, 1, 2, 4)$

If $\text{rank}_T(u) \leq r$ then $\text{rank}_{T_\sigma}(u)$ typically depends on $d$.

- Consider the Henon-Heiles potential

$$u(x) = \frac{1}{2}\sum_{i=1}^{d} x_i^2 + 0.2\sum_{i=1}^{d-1}(x_i x_{i+1}^2 - x_i^3) + \frac{0.2^2}{16}\sum_{i=1}^{d-1}(x_i^2 + x_{i+1}^2)^2$$

Using a linear tree $T = \{\{1\},\{2\},\ldots,\{d\},\{1,2\},\{1,2,3\},\ldots,\{1,\ldots,d-1\},D\}$,

$$\text{rank}_T(u) \leq 4, \quad \textit{storage}(u) = O(d)$$

but for the permutation

$$\sigma = (1,3,\ldots,d-1,2,4,\ldots,d) \qquad (\star)$$

and the corresponding linear tree $T_\sigma$,

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- For a typical tensor in $\mathcal{T}_r^T$ with $T$ a binary tree, its representation in tree based format with tree $T_\sigma$, with $\sigma$ as in $(\star)$, has a complexity scaling exponentially with $d$.
- As an example, consider the function $u(x,t) = 1_{x+t<0}$identified (through tensorization) with tensors $\boldsymbol{u}(i_1^x,\ldots,i_L^x,y^x,i_1^t,\ldots,i_L^t,y^t)$ and $\boldsymbol{u}(i_1^x,i_1^y,\ldots,i_L^x,i_L^y,y^x,y^t)$. Huge impact of the ordering !

- Consider the probability distribution $f(x) = \mathbb{P}(X = x)$ of a Markov chain $X = (X_1, \ldots, X_d)$ given by

$$f(x) = f_1(x_1) f_{2|1}(x_2|x_1) \ldots f_{d|d-1}(x_d|x_{d-1})$$

where bivariate functions $f_{i|i-1}$ have a rank $r$.

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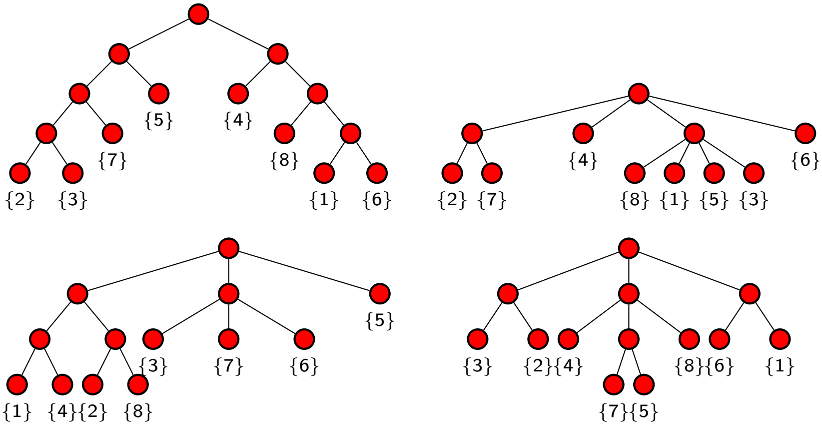
  - With the linear tree $T$ containing interior nodes $\{1,2\}, \{1,2,3\}, \dots, \{1, \dots, d-1\}$, $f$ admits a representation in tree-based format with storage complexity in $r^4$.
  - The canonical rank of $f$ is exponential in $d$.
  - But when considering the linear tree $T_\sigma$ obtained by applying permutation $\sigma = (1, 3, \dots, d-1, 2, 4, \dots, d)$ to the tree $T$, the storage complexity in tree-based format is also exponential in $d$.

A combinatorial problem...



# Outline

# Properties of tree tensor networks

We here consider approximation tools $(\Phi_n)_{n\geq 1}$ based on tensorization and tensor train format (with or without sparsity).
They satisfy

(P1) $\Phi_0 = \{0\}$, $0 \in \Phi_n$

(P2) $a\Phi_n = \Phi_n$ for any $a \in \mathbb{R} \setminus \{0\}$ (cone)

(P3) $\Phi_n \subset \Phi_{n+1}$ (nestedness)

(P4) $\Phi_n + \Phi_n \subset \Phi_{cn}$ for some constant $c$ (not too nonlinear)

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For $X = L^p$, they further satisfy

(P5) $\bigcup_n \Phi_n$ is dense in $L^p$ for $0 < p < \infty$ (universality),

(P6) for each $f \in L^p$ for $0 < p \leq \infty$, there exists a best approximation in $\Phi_n^{\mathcal{F}}$ (proximinal sets). However, $\Phi_n^{\mathcal{S}}$ is not closed.

# Approximation classes

For an approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$, we define for any $\alpha > 0$ the approximation class

$$A^\alpha_\infty(L^p) := A^\alpha_\infty(L^p, \Phi)$$

of functions $f \in L^p$ such that

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- Full and sparse complexity measures yield two different approximation spaces

$$\mathcal{F}^\alpha_\infty(L^p) = A^\alpha_\infty(L^p, \Phi^{\mathcal{F}}), \quad \mathcal{S}^\alpha_\infty(L^p) = A^\alpha_\infty(L^p, \Phi^{\mathcal{S}})$$

  such that

$$\mathcal{F}^\alpha_\infty(L^p) \hookrightarrow \mathcal{S}^\alpha_\infty(L^p) \hookrightarrow \mathcal{F}^{\alpha/2}_\infty(L^p)$$

# Direct embeddings

From results on the approximation properties for Besov spaces, we have the following results.

- (Linear approximation) For $\alpha > 0$ and $0 < p \leq \infty$,

$$B_q^\alpha(L^p) \hookrightarrow \mathcal{F}_\infty^{\alpha/d}(L^p),$$

$$MB_q^\alpha(L^p) \hookrightarrow \mathcal{S}_\infty^{\alpha}(L^p),$$

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- (Nonlinear approximation) For $\alpha > 0$, $1 \leq p < \infty$, $0 < q \leq \tau < p < \infty$ and $\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}$,

$$B_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p),$$

$$MB_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/2}(L^p)$$

  for arbitrary $\tilde{\alpha} < \alpha$, and

$$AB_q^{\boldsymbol{\alpha}}(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p)$$

  for arbitrary $\tilde{\alpha} < s(\boldsymbol{\alpha})$.

# Interpolation family

The properties of $\Phi_n$ allow to apply classical results from approximation theory, in particular to deduce from embedding results on $A^{\alpha}_{\infty}(L^p)$ embedding results on interpolation spaces

$$A^{\beta}_{q}(L^p) = (L^p, A^{\alpha}_{\infty}(L^p))_{\beta/\alpha,q}, \quad 0 < \beta < \alpha, \qquad 0 < q \leq \infty$$

that are quasi-Banach spaces with quasi-norm

$$\|f\|_{A^{\alpha}_{q}} = \|f\|_{L^p} + |f|_{A^{\alpha}_{q}}, \quad |f|_{A^{\alpha}_{q}} = \left(\sum_{n=1}^{\infty} n^{-1} \left(n^{\alpha} E(f, \Phi_n)_X\right)^q\right)^{1/q}$$

(functions with faster convergence than those of $A^{\alpha}_{\infty}(L^p)$).

# No inverse embedding

For any $\alpha > 0$, $q \leq \infty$, and any $\beta$,

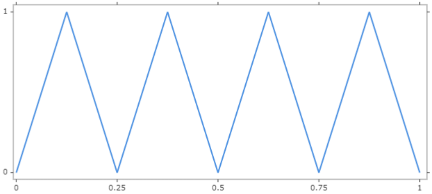
$$\mathcal{F}_q^\alpha(L^p) \not\hookrightarrow B_q^\beta(L^p).$$

That means that approximation classes contain functions that have no smoothness in a classical sense.

Tree tensor networks may be useful for the approximation of functions beyond standard smoothness classes.

# No inverse embedding

This is proved by contradiction by considering the sawtooth function $\varphi_L$ with $2^L$ teeth such that $\varphi_L \in \Phi_n$ with $n \sim L$.




From properties (P1)-(P6), $\mathcal{F}_q^\alpha(L^p)$ satisfies the Berstein inequality, that is

$$\|\varphi\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha \|\varphi\|_{L^p} \quad \forall \varphi \in \Phi_n.$$

Moreover, $\|\varphi_L\|_{L^p} \sim 1$ and $\|\varphi_L\|_{B_q^\beta(L^p)} \gtrsim 2^{\beta L}$. If the embedding were true, we would have

$$2^{\beta n} \lesssim \|\varphi_L\|_{B_q^\beta(L^p)} \lesssim \|\varphi_L\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha,$$

a contradiction.

# The role of depth

Consider the approximation with restricted resolution

$$\Phi_n^{\mathcal{L}} = \{f \in \Phi_n : L(f) \leq \mathcal{L}(n)\}$$

where $L(f)$ is the minimal resolution $L$ such that $f \in V_L$, and $\mathcal{L}$ some growth function.

Since $L(f) \leq n$ for $f \in \Phi_n$, $\Phi_n^{\mathcal{L}} = \Phi_n$ for $\mathcal{L} = n$.

In dimension $d = 1$, for $\mathcal{L}(n) = r \log_b(n) + c$, the following Bernstein inequality holds

$$|f|_{B_\tau^{m+1}(L^\tau)} \lesssim \|f\|_{L^p} b^{c(m+1)} n^{r(m+1)}$$

with $\tau$ the Sobolev embedding number, and $m$ the local polynomial degree. This implies the inverse embedding of the corresponding approximation class

$$A_\infty^\alpha(L^p; (\Phi_n^{\mathcal{L}})) \hookrightarrow B_\tau^{\alpha/(m+1)}(L^\tau)$$

Hence the importance of depth $L$ for going beyond standard regularity classes.

# Overview of results on approximation theory of tensor networks

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- For Sobolev spaces $W^{\alpha,p}$ or Besov spaces $B^{\alpha}_{q}(L^{p})$, tensor networks (full or sparse) achieve optimal rate in $O(n^{-\alpha/d})$.
- For Besov spaces $B^{\alpha}_{q}(L^{\tau})$ $(\tau < p)$, sparse tensor networks achieve arbitrary close to optimal rate in $O(n^{-\alpha/d})$, while full tensor networks achieve a rate arbitrarily close to $O(n^{-\alpha/(2d)})$.

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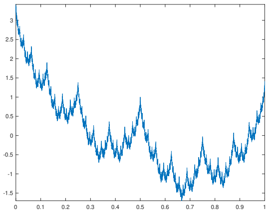
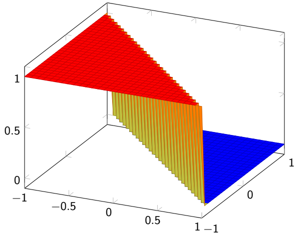
- Curse of dimensionality can be circumvented for non usual function classes such as compositions of smooth functions

- Can approximate very efficiently functions with low (or no) regularity in a usual sense: discontinuous functions, nowhere differentiable functions, fractals



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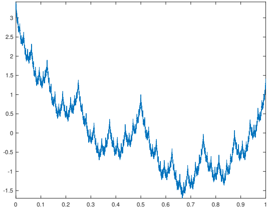


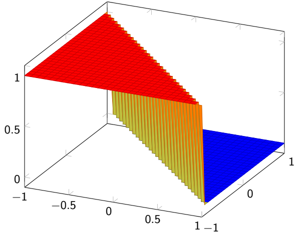


- Approximation classes of tensor networks (using tensorization) are not embedded in any Besov space. Tensor networks can efficiently approximate functions beyond standard smoothness classes.

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- The choice of the tensor format (ordering of variables, dimension partition tree $T$) may have a big influence on the performance.

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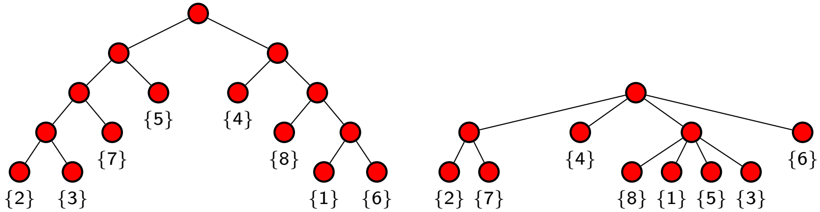
$$f(x) = f_1(x_1) f_{2|1}(x_2|x_1) \dots f_{d|d-1}(x_d|x_{d-1})$$

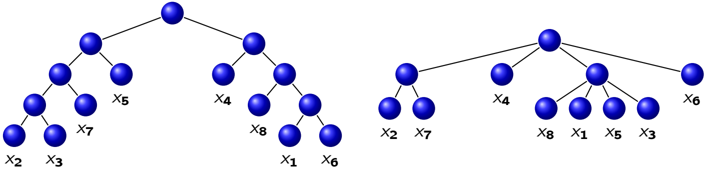
- How to select a good tree ? Combinatorial problem. Possible stochastic algorithms.



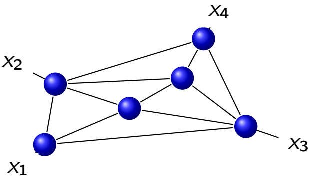
## Some open questions

- What are the properties of the approximation tool with free tree $T$ over $\{1, \dots, (L+1)d\}$

$$\Phi_n = \{f \in \Phi_{L,T,r,m} : L \in \mathbb{N}_0,\ T \subset 2^{\{1,\dots,(L+1)d\}}, r \in \mathbb{N}^{\#T}, compl(f) \le n\} \quad ?$$




- What about approximation classes of more general tensor networks ?



# Some open questions

- Algorithms to practically compute approximations achieving a certain precision with almost optimal complexity, using available information on the function (model equations, point samples...)
- Computational complexity of (deterministic or randomized) algorithms based on point samples for functions from approximation classes of tensor networks ?
- Theory to practice gap ?

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