GRAN SASSO Science Institute
Intensive Trimester "Particles, Fluids and Patterns: Analytical and Computational Challenges"
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High-dimensional approximation and sampling

Part 2: High-dimensional approximation tools

Anthony Nouy

Centrale Nantes, Nantes Université, Laboratoire de Mathématiques Jean Leray

Approximation tools for high-dimensional approximation

We here present classical approximation tools (model classes) for the approximation of multivariate functions

$$u(x_1, \ldots, x_d)$$

Outline

Expansions on tensor product bases

- Polynomials

$$\sum_{\alpha \in \Lambda} a_\alpha x^\alpha, \quad x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$$

where $\Lambda \subset \mathbb{N}^d$ is a set of multi-indices, either fixed (linear approximation) or free (nonlinear approximation).

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- More general expansions on tensorized bases

$$\sum_{\alpha \in \Lambda} a_\alpha \psi_\alpha(x), \quad \psi_\alpha(x) = \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_d}(x_d),$$

  e.g. tensor product splines, wavelets...

Additive and multiplicative models

- Additive models

$$u_1(x_1) + \ldots + u_d(x_d)$$

or more generally

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- Multiplicative models

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or more generally

$$\prod_{\alpha \in \Lambda} u_\alpha(x_\alpha)$$

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Separation of variables and tensor networks

- Sum of multiplicative models (canonical tensor format)

$$\sum_{k=1}^{r} v^{(1)}(x_1, k) \dots v^{(d)}(x_d, k)$$

that is a r-term approximation from the dictionary of separated functions.

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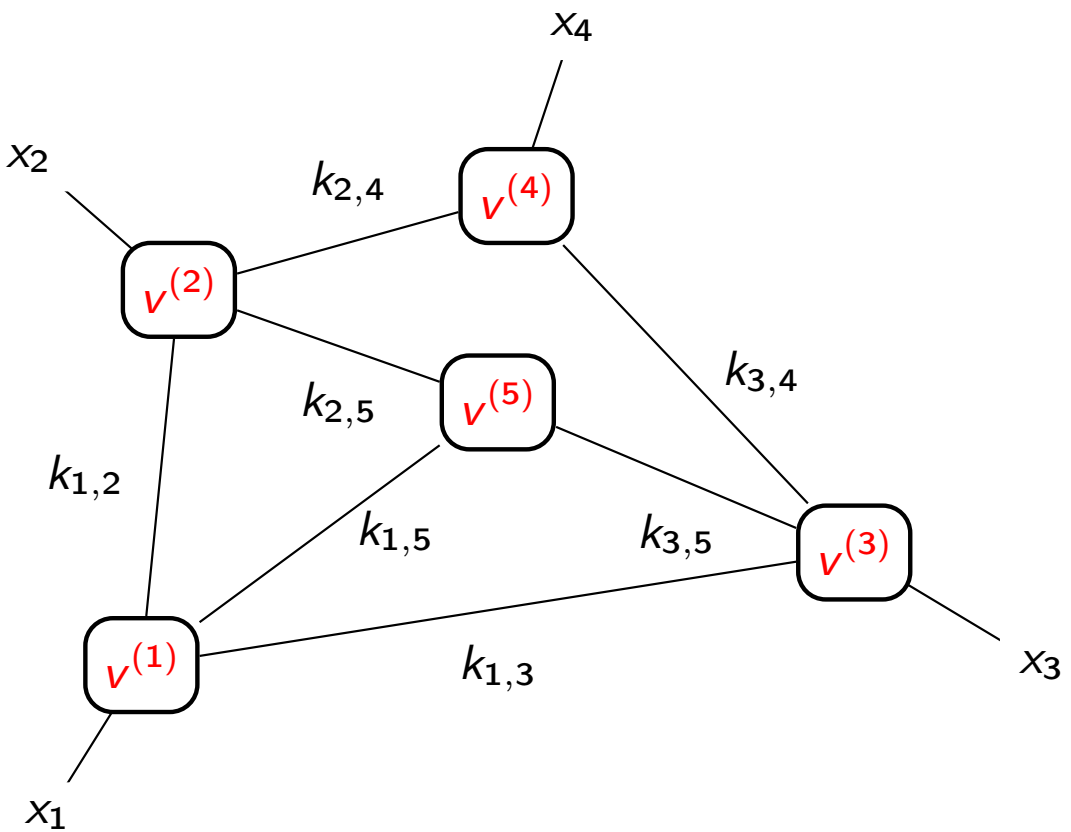
- Tensor train (Matrix Product State)

$$v(x) = \sum_{k_1=1}^{r_1} \dots \sum_{k_{d-1}=1}^{r_{d-1}} v^{(1)}(x_1, k_1) v^{(2)}(k_1, x_2, k_2) \dots v^{(d)}(k_{d-1}, x_d).$$

It is a particular case of tensor networks.

Separation of variables and tensor networks

- Tensor networks associated with general graphs



Composition of functions

$$f(g(x))$$

with $g : \mathbb{R}^d \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$.

The map g extracts m features $g(x)$ (new variables) from an input x. It can be fixed (application-dependent) or free.

For linear maps $g(x) = Ax$, this corresponds to

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Different regimes:

- small m ($m < d$), g performs a dimension reduction and f is a low-dimensional function. This corresponds to ridge approximation for $m = 1$.
  Various approaches to estimate A, including active subspace methods (derivative-based) [Constantine et al 2014, Zahm et al 2020].
  See [Bigoni et al 2022, Nouy and Pasco 2025] for extensions to nonlinear maps g.

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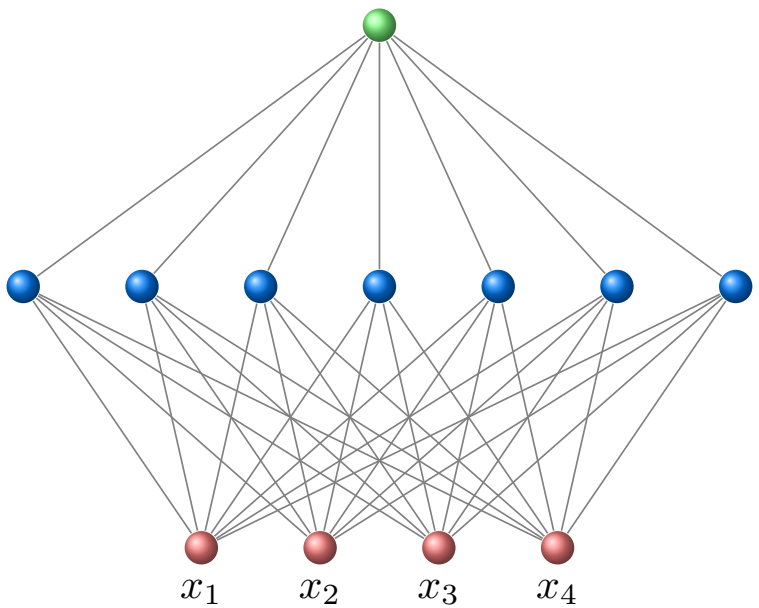
  See [Bigoni et al 2022, Nouy and Pasco 2025] for extensions to nonlinear maps g.
- large m, g extracts many features and f is expected to be simple, e.g. linear or additive.

Shallow neural networks

A shallow neural network (with one hidden layer of width m) is a sum of ridge functions

$$f(Ax) = a^T \sigma(Ax + b) = \sum_{i=1}^{m} a_i \sigma(\sum_{j=1}^{d} A_{ij} x_j + b_i)$$

where σ is a given function (activation function). Here f is an additive function.



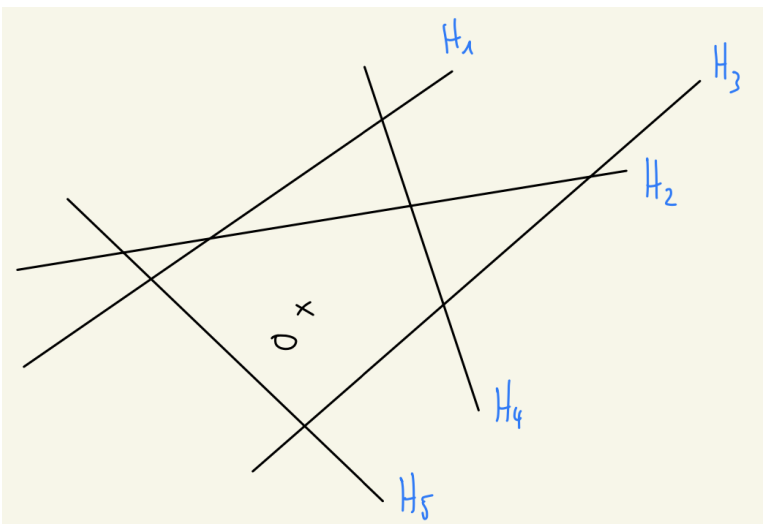
Shallow neural networks

Classical piecewise polynomial activation functions

- ReLU function $\sigma(t) = \langle t \rangle_+ = \max\{0, t\}$
- RePU(k) function $\sigma(t) = \langle t \rangle_+^k = \max\{0, t\}^k$

ReLU and RePU networks produce a piecewise polynomial approximation (spline) on a free partition of $\mathbb{R}^d$ determined by m hyperplanes

$$H_i = \{x : w_i{}^T x + b_i = 0\}, \quad w_i = (A_{ij})_{j=1}^d \in \mathbb{R}^d$$



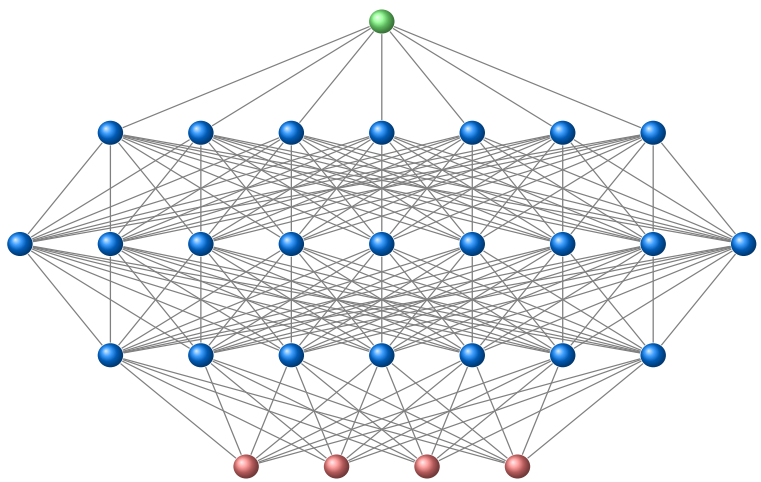
Deep neural networks

$$T_L \circ \sigma \circ T_{L-1} \circ \ldots \circ T_1 \circ \sigma \circ T_0(x)$$

with $T_\ell : \mathbb{R}^{m_\ell} \to \mathbb{R}^{m_{\ell+1}}$ an affine linear map

$$T_\ell(x) = A_\ell x + b_\ell$$

and $(m_1, \ldots, m_L) \in \mathbb{N}^L$ with $m_0 = d$, $m_{L+1} = 1$.



For ReLU or RePU(k) activation function σ, the approximation is a piecewise polynomial on a free partition with a number of domains growing exponentially with depth L.

Outline

Approximation tools based on neural networks

Different approximation tools $(X_n)_{n\geq 1}$ can be defined depending on which parameters are free (possible architectures) and how complexity is measured.

Letting $\Phi_{L,m}$ be the class of neural networks with depth L and widths $m = (m_1, \ldots, m_L)$, we define

$$X_n = \{v \in \Phi_{L,m} : L \in \mathcal{L}, m \in \mathcal{M}_L, compl(v) \leq n\}$$

where *compl* is a complexity measure, $\mathcal{L} \subset \mathbb{N}$ is the set of possible depths and $\mathcal{M}_L \subset \mathbb{N}^L$ the set of possible widths.

Two typical classes of architectures

- Fixed depth L and free width:

$$\mathcal{L} = \{L\}, \quad \mathcal{M}_L = \{(W, \ldots, W) : W \in \mathbb{N}\}$$

- Free depth and fixed width W:

$$\mathcal{L} = \mathbb{N}, \quad \mathcal{M}_L = \{(W, \ldots, W)\}$$

Approximation tools based on neural networks

For a function v in the class $\Phi_{L,m}$ of neural networks with depth L and widths $m = (m_1, \ldots, m_L)$, different measures of complexity:

- number of parameters (fully connected networks)

$$compl_F(v) = \sum_{\ell=0}^{L} m_\ell m_{\ell+1} + m_{\ell+1} \sim W^2 L \text{ for } m_\ell \sim W$$

- number of non-zero parameters (sparsely connected networks)

$$compl_S(v) = \sum_{\ell=0}^{L} \|A_\ell\|_0 + \|b_\ell\|_0$$

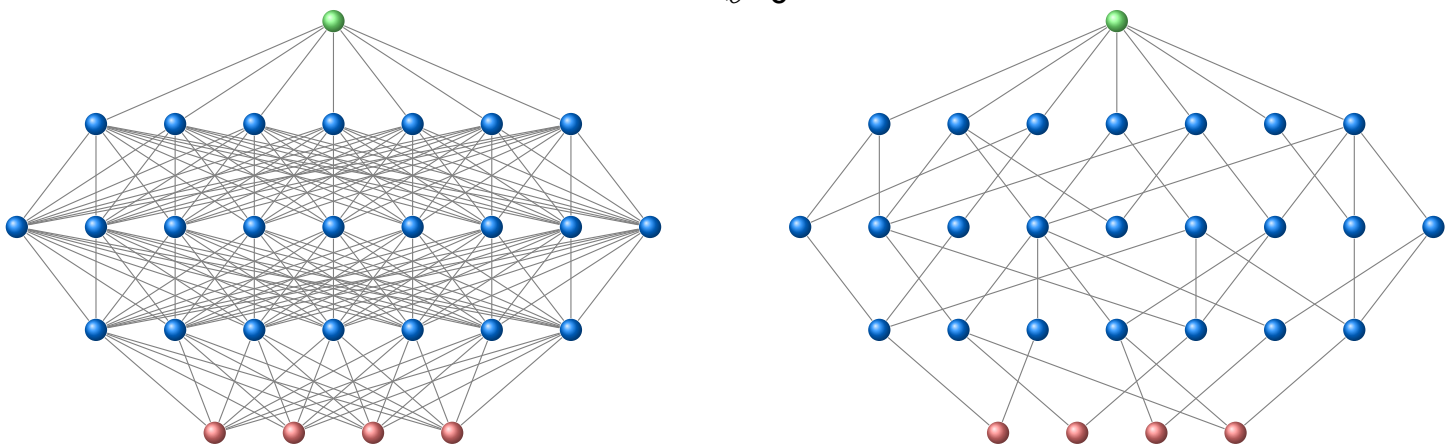


Figure: Fully connected network (left) and Sparsely connected network (right).

Structured sparsity can be imposed (convolutional NN, recurrent NN...) or sparsity pattern can be considered as a free parameter (a challenge on the algorithmic side).

Deep neural networks approximation theory

Many recent results on the expressivity of (deep) neural networks for various model classes [Petersen and Zech 2025].

- Shallow neural networks with RePU(k) activation function achieve an optimal rate in L^p-norm ($p \geq 2$) in $O(n^{-s/d})$ for functions in $W^{s,p}((0,1)^d)$, for any $s \leq k + 1 + (d-1)/2$ [Mao et al 2024].

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  E.g. for functions u in $W^{s,\infty}((0,1)^d)$, ReLU networks achieve

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- Approximation classes of deep ReLU networks are not embedded in standard smoothness classes [Gribonval et al 2021].
- They approximate efficiently functions beyond smoothness classes (discontinuous functions, fractals, refinable functions...).

Deep neural networks approximation theory

A few surprises

- For functions u in the unit ball K of $W^{s,\infty}((0,1)^d)$, ReLU networks with free depth can achieve

$$e_n(u)_{L^\infty} \lesssim n^{-q} \quad \text{for arbitrary } q \leq 2s/d.$$

  However, since the manifold width $\delta_n(K)_{L^\infty} \gtrsim n^{-s/d}$, a rate $q > s/d$ can be achieved only with discontinuous parameter selection. Also, it requires an encoding of parameters with more than $O(\log_2(\epsilon^{-1}))$ bits to achieve accuracy ϵ.

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Open problems

- Characterize functions that can be approximated stably with deep networks.
- Characterize functions that can be estimated with partial information and near optimal performance.
- Provide algorithms that achieve near to optimal performance.

Outline

Tensor networks

Tensor networks are prominent tools for the representation or approximation of multivariate functions or multidimensional arrays.

- A long history in quantum physics. Several tools exploiting separation of variables (MPS, PEPS, MERA...)
- Tree tensor networks (Hierarchical Tucker tensors) appeared independently in numerical analysis and numerical linear algebra, as an extension of low-rank decompositions to high-order tensors [Hackbusch and Kuhn, Grasedyck, Oseledets and Tyrtyshnikov].
- Growing use in statistics, data science and probabilistic modelling.

Tensor product of functions

Let $V_\nu \subset \mathbb{R}^{\mathcal{X}_\nu}$ be a space of functions defined on $\mathcal{X}_\nu$.

$\mathcal{X}_\nu$ can be (a subset of) $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, or a set of vectors, sequences, graphs, images...

The tensor product of functions $v^{(\nu)} \in V_\nu$, denoted

$$v = v^{(1)} \otimes \ldots \otimes v^{(d)},$$

is a multivariate function defined on $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d$ and such that

$$v(x_1, \ldots, x_d) = v^{(1)}(x_1) \ldots v^{(d)}(x_d)$$

Tensor product of functions

The algebraic tensor product of spaces V_ν is defined as

$$V_1 \otimes \ldots \otimes V_d = \text{span}\{v^{(1)} \otimes \ldots \otimes v^{(d)} : v^{(\nu)} \in V_\nu, 1 \leq \nu \leq d\}$$

which is the space of multivariate functions v which can be written as a finite linear combination of elementary (separated functions), i.e.

$$v(x_1, \ldots, x_d) = \sum_{k=1}^{r} v_k^{(1)}(x_1) \ldots v_k^{(d)}(x_d).$$

Rank of multivariate functions and canonical format

The canonical rank of a multivariate function $f(x_1, \ldots, x_d)$ is the minimal integer such that f has a representation

$$f(x) = \sum_{k=1}^{r} v_k^1(x_1) \ldots v_k^d(x_d)$$

Given a finite-dimensional tensor space $V = V_1 \otimes \ldots \otimes V_d$ of multivariate functions we define a canonical tensor format in V as a set of functions

$$\mathcal{R}_r(V) = \{f \in V : \text{rank}(f) \leq r\}$$

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From the practical point of view, it is not a nice format. In particular, $\mathcal{R}_r(V)$ is not closed for $d \geq 3$ and $r \geq 2$.

For any continuous parametrization $\mathcal{R}_r(V) = \{v = R(p) : p \in P\}$, and for any tensor of $v \in \overline{\mathcal{R}_r(V)} \setminus \mathcal{R}_r(V)$ of border rank r, the quantity

$$\delta(v, \epsilon) = \inf\{\|p\| : \|v - R(p)\| < \epsilon\}$$

diverges as $\epsilon \to 0$ [Hackbusch 2021].

α-ranks of multivariate functions

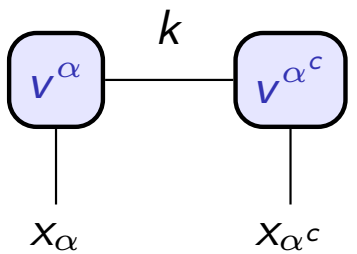
A multivariate function $f(x_1, \ldots, x_d)$, for any set $\alpha \subset \{1, \ldots, d\}$, can be identified with a bivariate function $f(x_\alpha, x_{\alpha^c})$ of two complementary subsets of variables.

The rank of the bivariate function $f(x_\alpha, x_{\alpha^c})$ is the α-rank of f, denoted $\text{rank}_\alpha(f)$.

A function with α-rank bounded by r_α admits a representation

$$f(x) = \sum_{k=1}^{r_\alpha} v_k^\alpha(x_\alpha) v_k^{\alpha^c}(x_{\alpha^c})$$

or using tensor diagram notations

$f(x) =$ 

where a connection between two tensors represents a contraction along one mode of each tensor.

Example

- $u(x) = u^1(x_1) \dots u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha) u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \prod_{\nu \in \alpha} u^\nu(x_\nu)$. Therefore, for any α, $\text{rank}_\alpha(u) = 1$.

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  We deduce the following relation between α-ranks and canonical rank:

  $$\text{rank}_\alpha(u) \leq \text{rank}(u), \quad \text{for any } \alpha.$$

- $u(x) = u^1(x_1) + \dots + u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha) + u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \sum_{\nu \in \alpha} u^\nu(x_\nu)$. Therefore, $\text{rank}_\alpha(u) \leq 2$.

- $u(x) = u^1(x_1, x_2) + u^2(x_2, x_3) \dots + u^{d-1}(x_{d-1}, x_d)$ has $\{1, \dots, k\}$-rank bounded by $\text{rank}(u^k) + 1$ for any $1 \leq k \leq d-1$.

Low-rank tensor format

Given

- a finite-dimensional tensor space $V = V_1 \otimes \ldots \otimes V_d$ of multivariate functions
- a collection T of subsets in $\{1, \ldots, d\}$,
- a tuple of ranks $r = (r_\alpha)_{\alpha \in T}$,

we define a low-rank tensor format in V as a set of functions

$$\mathcal{T}_r^T(V) = \{f \in V : \mathrm{rank}_\alpha(f) \leq r_\alpha, \alpha \in T\}$$

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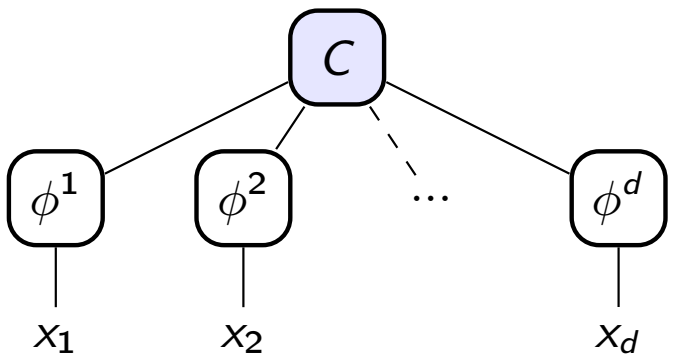
with representation

$$f(x) = \sum_{i_1 \in I_1} \ldots \sum_{i_d \in I_d} C(i_1, \ldots, i_d) \phi^1(x_1)_{i_1} \ldots \phi^d(x_d)_{i_d} =$$



where ϕ^ν is a feature map associated with V^ν and $C \in \mathbb{R}^{I_1 \times \ldots \times I_d}$ is a rank-structured algebraic tensor.

Tensor train format

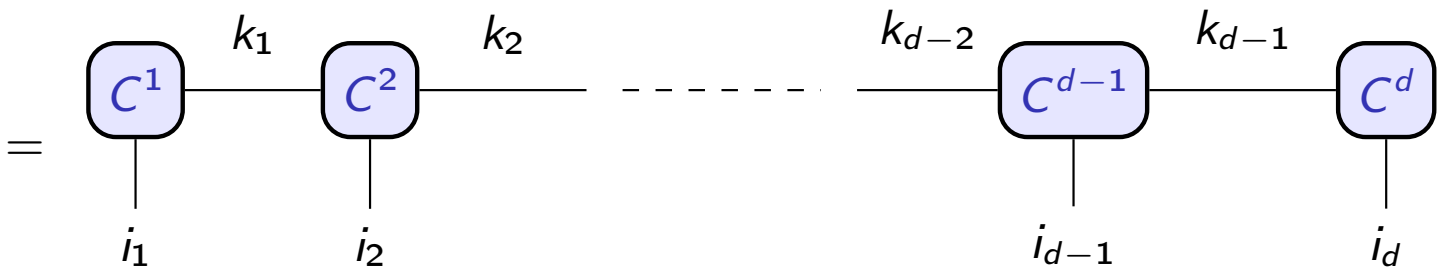
With

$$T = \{\{1\}, \{1,2\}, ..., \{1,\ldots,d\}\},$$

$\mathcal{T}_r^T(V)$ coincides with the tensor train format.

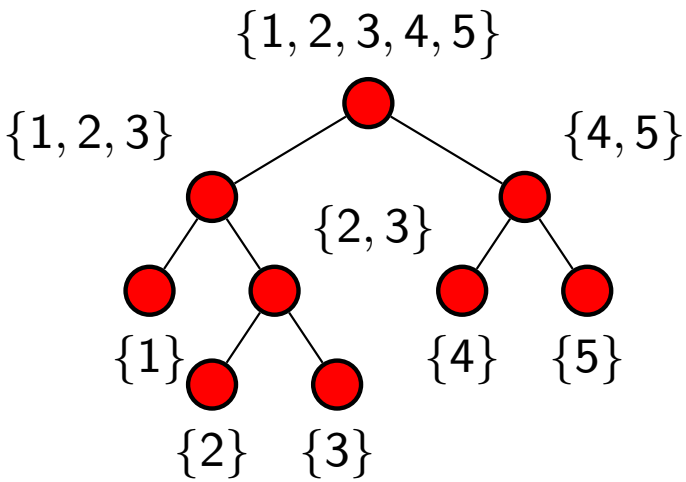
A function f in $\mathcal{T}_r^T(V)$ has coefficients

$$C(i_1,\ldots,i_d) = \sum_{k_1=1}^{r_1} \ldots \sum_{k_{d-1}=1}^{r_{d-1}} C^1(i_1,k_1)C^2(k_1,i_2,k_2)\ldots C^d(k_{d-1},i_d).$$

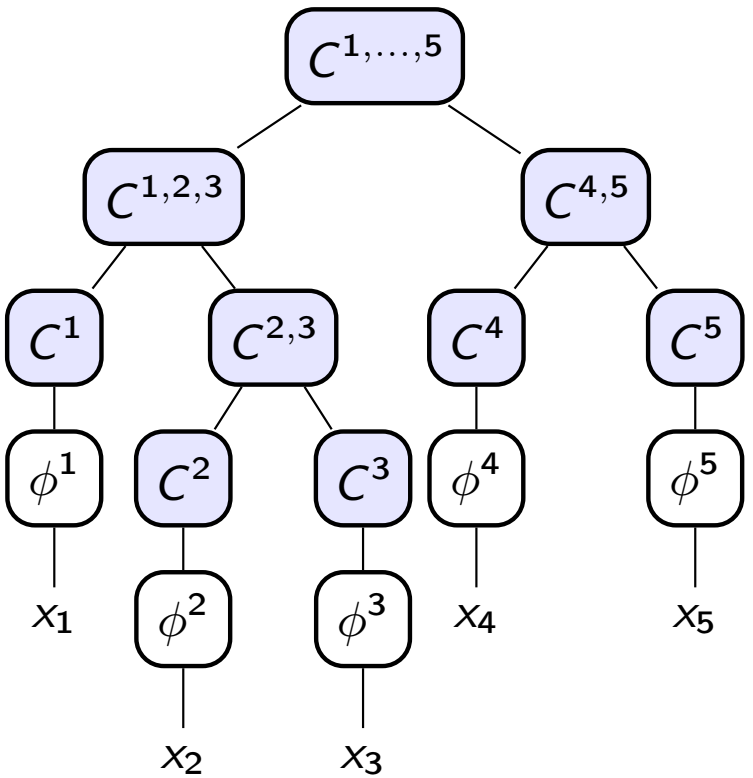


Hierarchical Tucker format (Tree tensor networks)

If T is a dimension partition tree, $\mathcal{T}_r^T(V)$ is a tree-based (or hierarchical) tensor format and a function in $\mathcal{T}_r^T(V)$ admits a multilinear parametrization with a collection of parameters $\{C^\alpha : \alpha \in T\}$ forming a tree tensor network.



Dimension tree T



Tree tensor network

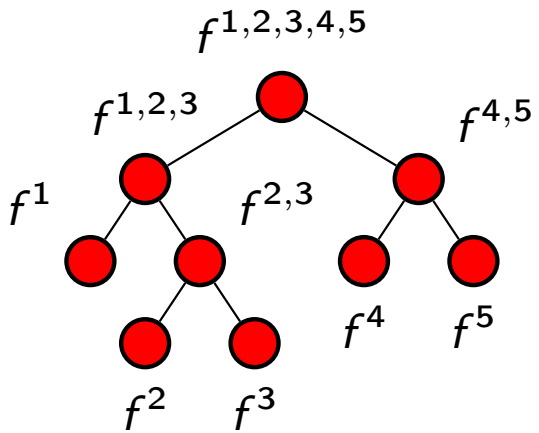
Tree tensor networks as a compositional function network

By identifying a tensor $C^{(\alpha)} \in \mathbb{R}^{n_1 \times \ldots \times n_s \times r_\alpha}$ with a $\mathbb{R}^{r_\alpha}$-valued multilinear function

$$f^{(\alpha)} : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s} \to \mathbb{R}^{r_\alpha},$$

a function v in $\mathcal{T}_r^T(V)$ admits a representation as a tree-structured composition of multilinear functions $\{f^{(\alpha)}\}_{\alpha \in T}$, e.g.

$$v(x) = f^D(f^{1,2,3}(f^1(\phi^1(x_1)), f^{2,3}(f^2(\phi^2(x_2)), f^3(\phi^3(x_3)))), f^{4,5}(f^4(\phi^4(x_4)), f^5(\phi^5(x_5))))$$



Tree tensor networks as a compositional function network

A multilinear map f^{α} can also be written

$$f^{\alpha}(z_1, \ldots, z_s) = A^{\alpha}\sigma(z_1, \ldots, z_d), \quad z_k \in \mathbb{R}^{n_k},$$

with a matrix

$$A^{\alpha} \in \mathbb{R}^{r_{\alpha} \times N}, \quad N = n_1 \ldots n_s$$

and a fixed multilinear function

$$\sigma(z_1, \ldots, z_s) = vec(z_1 \otimes \ldots \otimes z_s) \in \mathbb{R}^N$$

Tree tensor networks as feed-forward neural networks

It corresponds to a sum-product feed forward neural network with a sparse architecture (given by T), a number of hidden layers equal to $depth(T) + 1$ (including a featuring layer), and width at level ℓ related to the α-ranks of the nodes α of level ℓ.

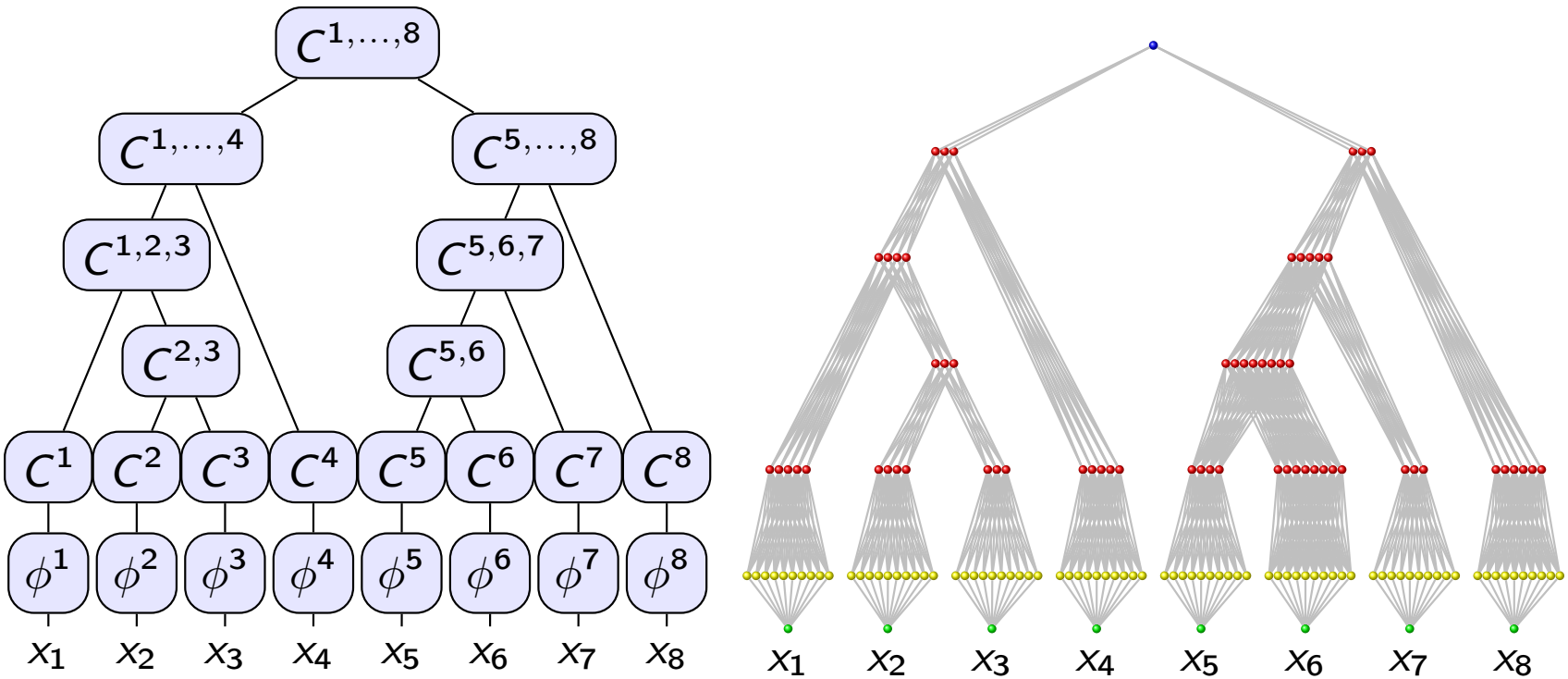


Figure: Tree tensor network and corresponding feed-forward sum-product neural network with 10 features per variable x_ν (right)

Approximation tools based on tree tensor networks

For the approximation of a function, a first approach is to introduce subspaces $V_{N_\nu}^\nu$ of finite dimension (e.g. polynomials, splines, wavelets, RKHS...) and consider tree tensor networks $f \in \mathcal{T}_r^T(V_N)$ where

$$V_N = V_{N_1}^1 \otimes \ldots \otimes V_{N_d}^d,$$

with variable $N = (N_1, \ldots, N_d)$ and r.

Spaces $V_{N_\nu}^\nu$ have to be well chosen, e.g. polynomials for analytic functions, splines with a degree adapted to the regularity of the function...

Approximation tools based on tree tensor networks

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \mathcal{T}_r^T(V_N) : N \in \mathbb{N}^d, r \in \mathbb{N}^T, \mathit{compl}(f) \leq n\}.$$

The dimensions N and the ranks r are free parameters, and $\mathit{compl}(\cdot)$ is some complexity measure.

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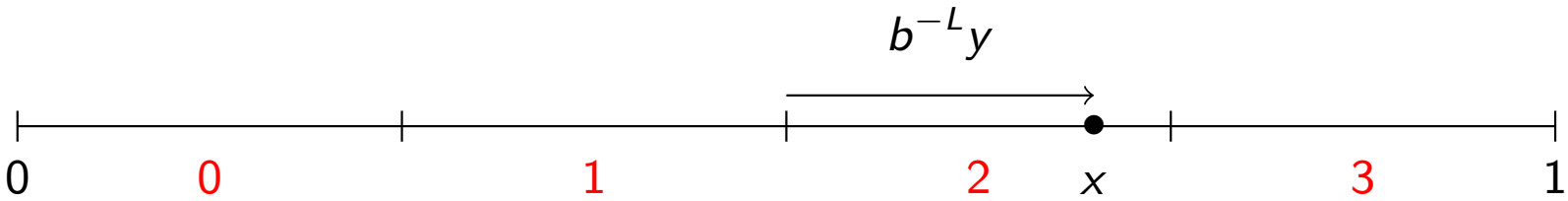
An alternative approach is to rely on tensorization of functions (specific featuring step).

Tensorization of univariate functions

Consider a function $f \in \mathbb{R}^{[0,1)}$ defined on the interval $[0,1)$.

- For $b, L \in \mathbb{N}$, we subdivide uniformly the interval $[0,1)$ into b^L intervals. Any $x \in [0,1)$ can be written

$$x = b^{-L}(i + y), \quad i \in \{0, \dots, b^L - 1\}, \quad y \in [0,1).$$



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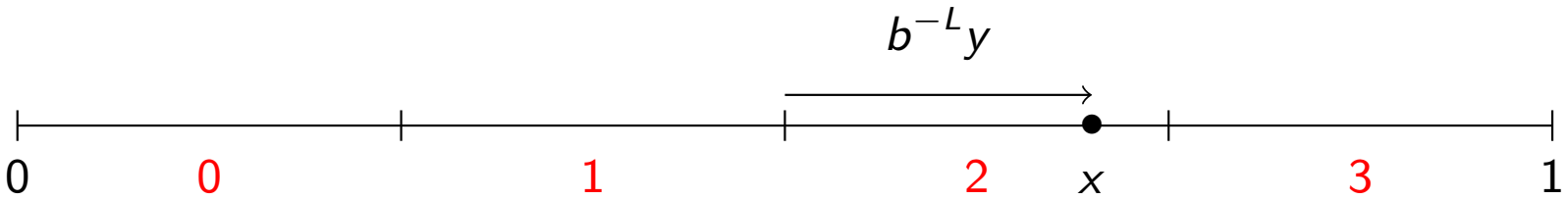
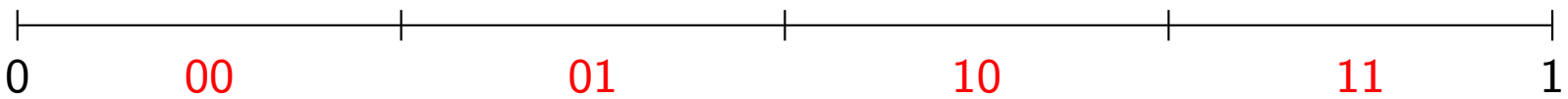


- The integer i admits a representation in base b

$$i = \sum_{k=1}^{L} i_k b^{L-k} = [i_1 \dots i_L]_b, \quad i_k \in \{0, \dots, b-1\}$$



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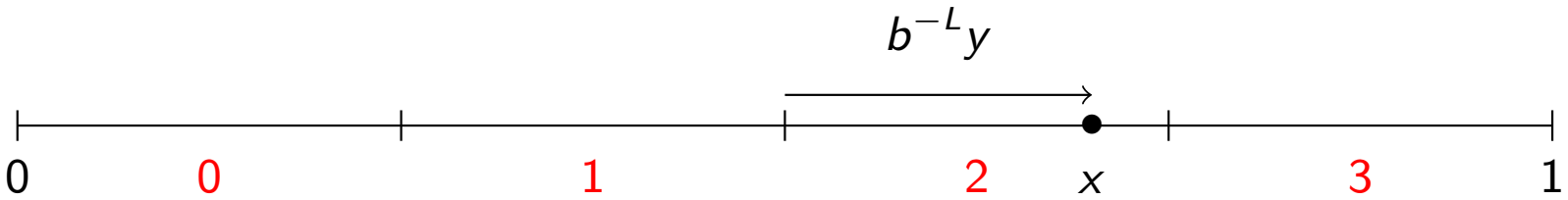

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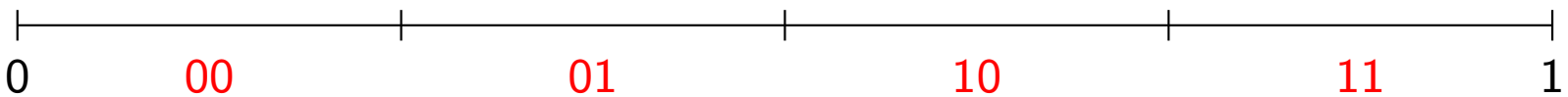
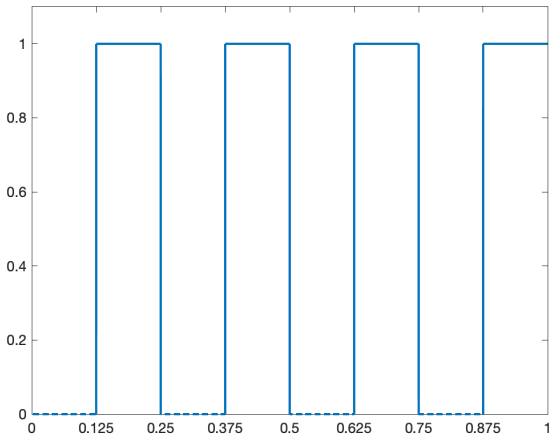


- f is thus identified with a multivariate function (tensor of order $L+1$)

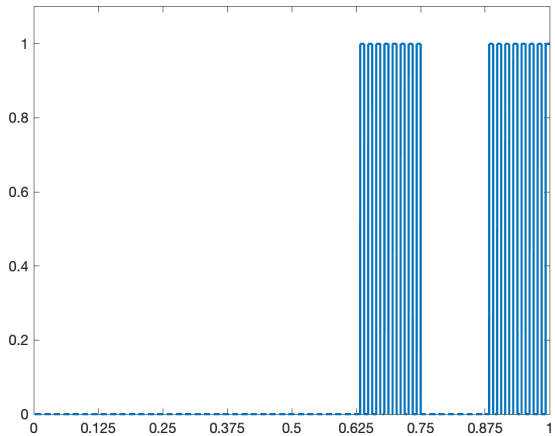
$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes L} \otimes \mathbb{R}^{[0,1)} \quad \text{such that} \quad f(x) = \boldsymbol{f}(i_1, \dots, i_L, y)$$

Tensorization of univariate functions

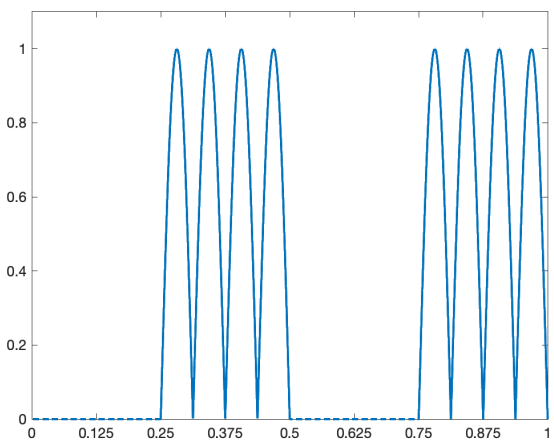
Digit $i_k(x)$ can be seen as a particular feature extracted from x.



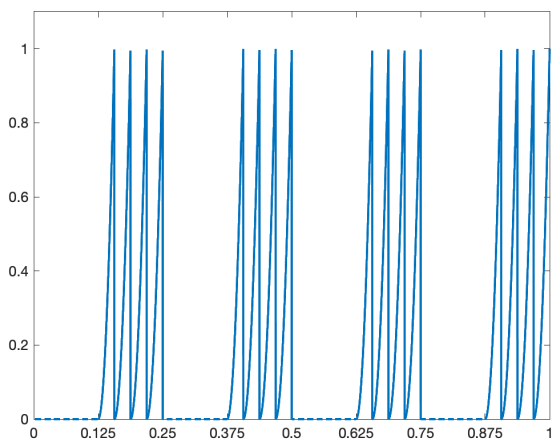
(a) $i_3(x)$



(b) $i_1(x)i_3(x)i_7(x)$



(c) $i_2(x)sin(\pi y)$ $(L = 4)$



(d) $i_3(x)y^2$ $(L = 5)$

Tensorization of multivariate functions

A function $f(x_1, \ldots, x_d)$ defined on $[0,1)^d$ can be similarly identified with a tensor of order $(L+1)d$

$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes Ld} \otimes (\mathbb{R}^{[0,1)})^{\otimes d}$$

such that

$$f(x_1, \ldots, x_d) = \boldsymbol{f}(i_1^1, \ldots, i_1^L, y_1, \ldots, i_d^1, \ldots, i_d^L, y_d)$$

$$\text{where} \quad x_\nu = b^{-L}\Big(\sum_{k=1}^{L} i_\nu^k b^{L-k} + y_\nu\Big) = [0.i_\nu^1 \ldots i_\nu^L]_b + b^{-L} y_\nu$$

Digits $(i_1^1, \ldots, i_1^L, \ldots, i_d^1, \ldots, i_d^L)$ encode a uniform partition of $[0,1)^d$ into b^{dL} elements.

0011	0111	1011	1111
0010	0110	1010	1110
0001	0101	1001	1101
0000	0100	1000	1100

Figure: d=2, b=2, L=2

Tensorization of multivariate functions

Using a different (resolution-wise) ordering of variables, the function can be identified with another tensor

$$f(x_1, \ldots, x_d) = \boldsymbol{f}(i_1^1, \ldots, i_d^1, \ldots, i_1^L, \ldots, i_d^L, y_1, \ldots, y_d)$$

It corresponds to another encoding of the partition of $[0,1)^d$ into b^{dL} elements.

0101	0111	1101	1111
0100	0110	1100	1110
0001	0011	1001	1011
0000	0010	1000	1010

Figure: d=2, b=2, L=2

Tensorization of multivariate functions

This particular re-parametrization is related to Morton space filling curve (or Z-order), which consists in mapping a point

$$([0.i_1^1 \dots i_1^L \dots]_2, \dots, [0.i_d^1 \dots i_d^L \dots]_2) \in [0,1]^d$$

to a real number

$$[0.i_1^1 \dots i_d^1 \dots i_1^L \dots i_d^L \dots]_2 \in [0,1]$$

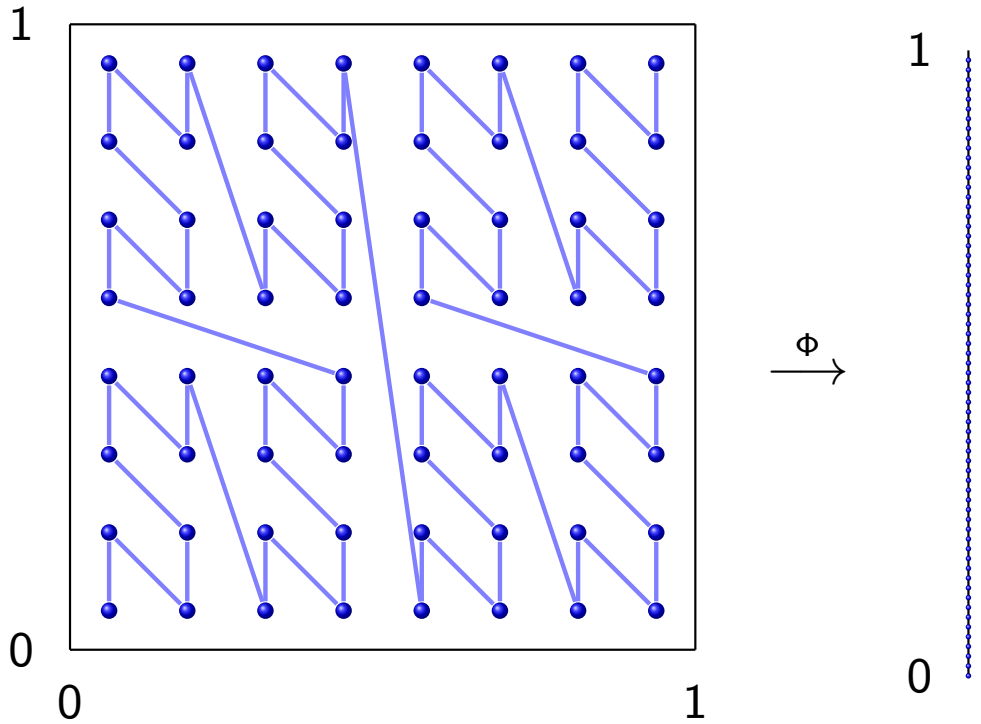
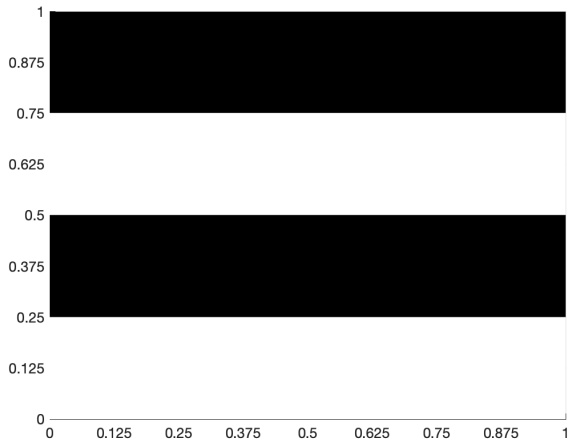


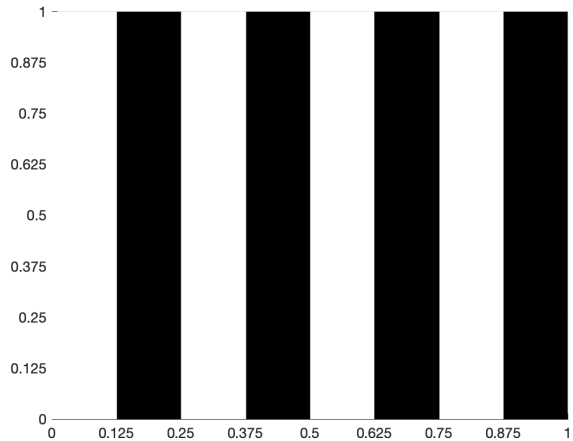
Figure: $b = 2$ and $L = 3$

Tensorization of multivariate functions

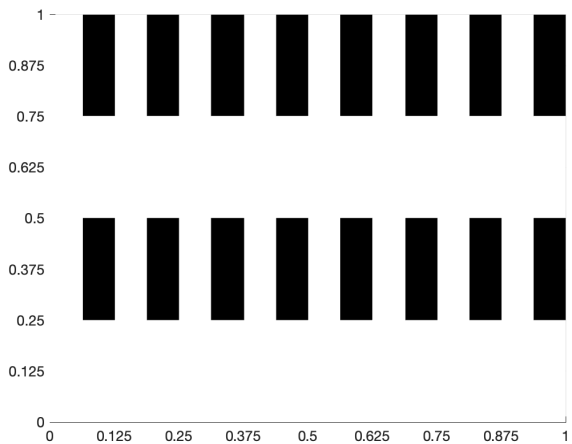
Digit $i_\nu^\ell(x)$ can be seen as a particular feature extracted from x.



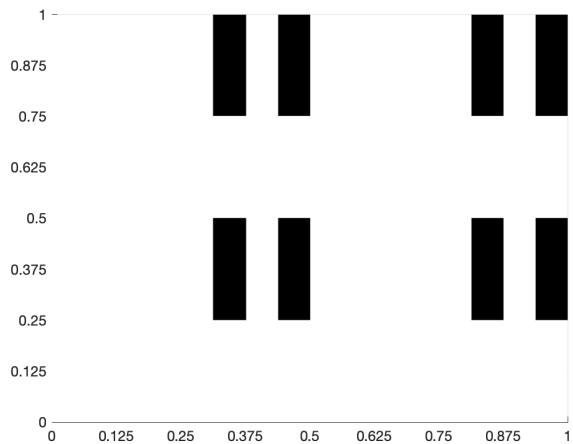
(a) $i_{\mathbf{1}}^{\mathbf{2}}(x)$



(b) $i_{\mathbf{2}}^{\mathbf{1}}(x)$



(c) $i_{\mathbf{1}}^{\mathbf{2}}(x)i_{\mathbf{3}}^{\mathbf{1}}(x)$



(d) $i_{\mathbf{1}}^{\mathbf{1}}(x)i_{\mathbf{1}}^{\mathbf{2}}(x)i_{\mathbf{3}}^{\mathbf{1}}(x)$

Tensorization of multivariate functions

The map $T_{b,L}$ which associates to a function f its tensorization $\boldsymbol{f}$ is a linear isometry from $L^p([0,1)^d)$ to $L^p(\{0,\dots,b-1\}^{Ld} \times [0,1)^d)$ for any $0 < p \leq \infty$.

Approximation tools based on tree tensor networks

We consider functions whose tensorization at resolution L are in the tensor space

$$\boldsymbol{V}_L = (\mathbb{R}^b)^{\otimes Ld} \otimes S^{\otimes d}$$

with $S \subset \mathbb{R}^{[0,1)}$ some subspace of univariate functions, invariant through b-adic dilation[1].

[1] S invariant through b-adic dilation means that $\psi \in S$ implies $\psi(b^{-1}(\cdot - k)) \in S$ for any $k \in \{0, \ldots, b-1\}$.

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If $S = \mathbb{P}_m$, $V_L = T_{b,L}^{-1}(\boldsymbol{V}_L)$ is identified with the space of multivariate splines of degree m over a uniform partition with b^{dL} elements, i.e.

$$V_L = V_{N_1}^1 \otimes \ldots \otimes V_{N_d}^d$$

with $N_1 = \ldots = N_d = b^L$ and $V_{N_\nu}^\nu$ a space of univariate splines of degree m over a uniform partition with $N_\nu = b^L$ intervals.

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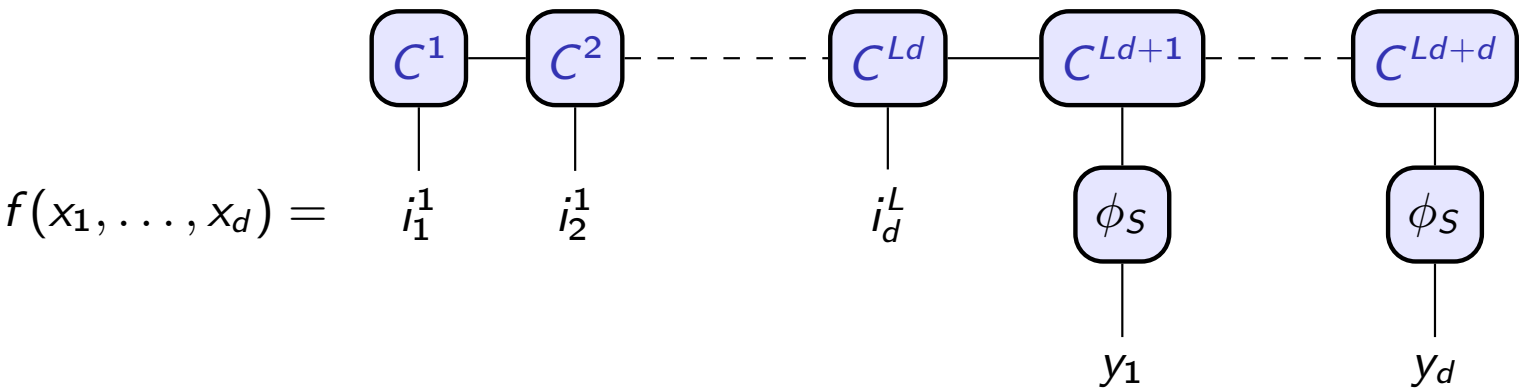
Approximation tools based on tree tensor networks

Then as an approximation tool, we consider functions f whose tensorization is a tensor network in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$, with T_L a dimension tree over $\{1, \ldots, Ld + d\}$.

Using the tensor train format, the corresponding function $f(x_1, \ldots, x_d)$ has the representation



with ϕ_S the feature map associated with S. This is closely related to the quantized tensor train (QTT) format [Kazeev, Khoromskij, Oseledets, Schwab, ...]

Later on, we consider $S = \mathbb{P}_m$ and $\phi_S(y) = (1, y, ..., y^{m+1})$ or any other polynomial basis.

Approximation tools based on tree tensor networks

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \Phi_{L,T_L,r} : L \in \mathbb{N}_0, r \in \mathbb{N}^{T_L}, \mathit{compl}(f) \leq n\}$$

with $\Phi_{L,T_L,r}$ the functions whose tensorization at resolution L is in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$.

The resolution L and ranks r are free parameters, and $\mathit{compl}(\cdot)$ is some complexity measure.

Complexity measures and corresponding approximation tools

The complexity $compl(f)$ of f is defined as the complexity of the associated tensor network $\{C^\alpha\}_{\alpha \in T}$.

- Number of parameters (full tensor network)

$$compl_{\mathcal{F}}(f) = \sum_\alpha \text{number\_of\_entries}(C^\alpha)$$

- Number of non-zero parameters (sparse tensor network)

$$compl_{\mathcal{S}}(f) = \sum_\alpha \|C^\alpha\|_0$$

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Complexity measures $compl_{\mathcal{F}}$ and $compl_{\mathcal{S}}$ yield two different approximation tools

$$\Phi_n^{\mathcal{F}} \quad \text{and} \quad \Phi_n^{\mathcal{S}}$$

such that

$$\Phi_n^{\mathcal{F}} \subset \Phi_n^{\mathcal{S}} \subset \Phi_{a+bn^2}^{\mathcal{F}}$$

Overview of results on approximation theory of tensor networks

See [Schneider and Uschmajew 2014, Ali and Nouy 2023, Ali and Nouy 2021, Bachmayr, Nouy and Schneider 2023, Kazeev and Schwab 2015].

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- For Besov spaces $B^{\alpha}_{q}(L^{\tau})$ ($\tau < p$), sparse tensor networks achieve arbitrary close to optimal rate in $O(n^{-\alpha/d})$, while full tensor networks achieve a rate arbitrarily close to $O(n^{-\alpha/(2d)})$.

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$$s(\boldsymbol{\alpha})/d = (\alpha_1^{-1} + \ldots + \alpha_d^{-1})^{-1}$$

the aggregated smoothness. Curse of dimensionality can be circumvented with sufficient anisotropy.

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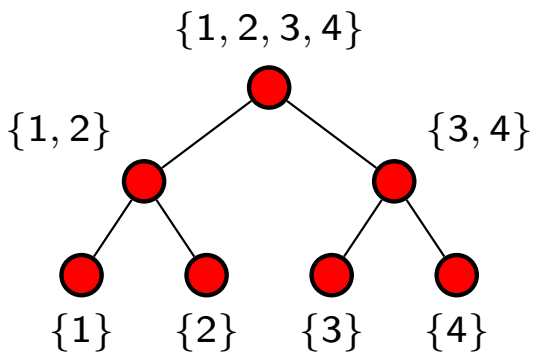
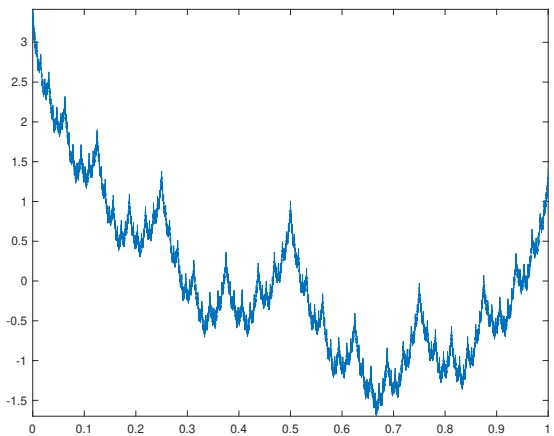
  the aggregated smoothness. Curse of dimensionality can be circumvented with sufficient anisotropy.
- Curse of dimensionality can be circumvented for non usual function classes such as compositions of smooth functions, provided the number of compositional layers is in $\log(d)^{\eta}$ or the functions that are composed are 1-Lipschitz [Bachmayr, Nouy and Schneider 2023].

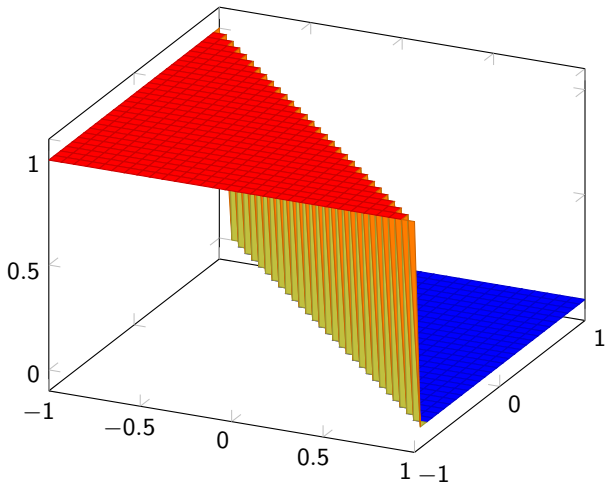
$$f_{1,2,3,4}\left(f_{1,2}\left(f_1(x_1), f_2(x_2)\right), f_{3,4}\left(f_3(x_3), f_4(x_4)\right)\right)$$



Overview of results on approximation theory of tensor networks

- Can approximate very efficiently functions with low (or no) regularity in a usual sense: discontinuous functions, nowhere differentiable functions, fractals





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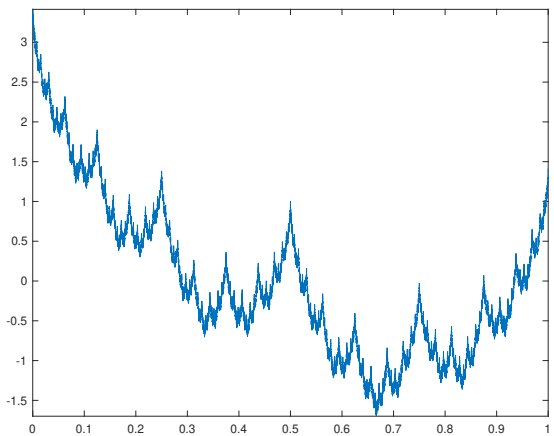

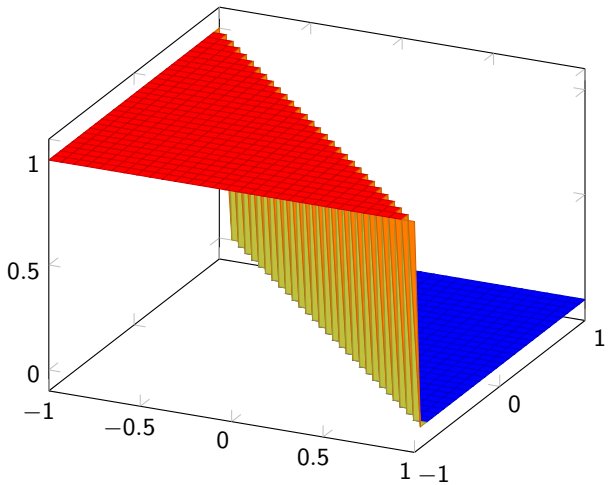

- Approximation classes of tensor networks (using tensorization) are not embedded in any Besov space. Tensor networks can efficiently approximate functions beyond standard smoothness classes.

Overview of results on approximation theory of tensor networks

- $\Phi_n^{\mathcal{F}}$ is closed while $\Phi_n^{\mathcal{S}}$ is not. Manipulating sparse tensor networks may be difficult in practice (to be compared with sparsely connected neural networks).

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- $\Phi_n^{\mathcal{F}}$ is closed while $\Phi_n^{\mathcal{S}}$ is not. Manipulating sparse tensor networks may be difficult in practice (to be compared with sparsely connected neural networks).
- Approximation classes of sparse tensor networks are larger than those of full tensor networks. However, if sparse tensor networks achieve a convergence in $O(n^{-\alpha})$, then full tensor networks achieve a convergence in $O(n^{-\alpha/2})$

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- Approximation classes of sparse tensor networks are larger than those of full tensor networks. However, if sparse tensor networks achieve a convergence in $O(n^{-\alpha})$, then full tensor networks achieve a convergence in $O(n^{-\alpha/2})$
- The choice of the tensor format (ordering of variables, dimension partition tree T) may have a big influence on the performance. For example, the function

$$f(x) = f_1(x_1) f_{2|1}(x_2|x_1) \dots f_{d|d-1}(x_d|x_{d-1})$$

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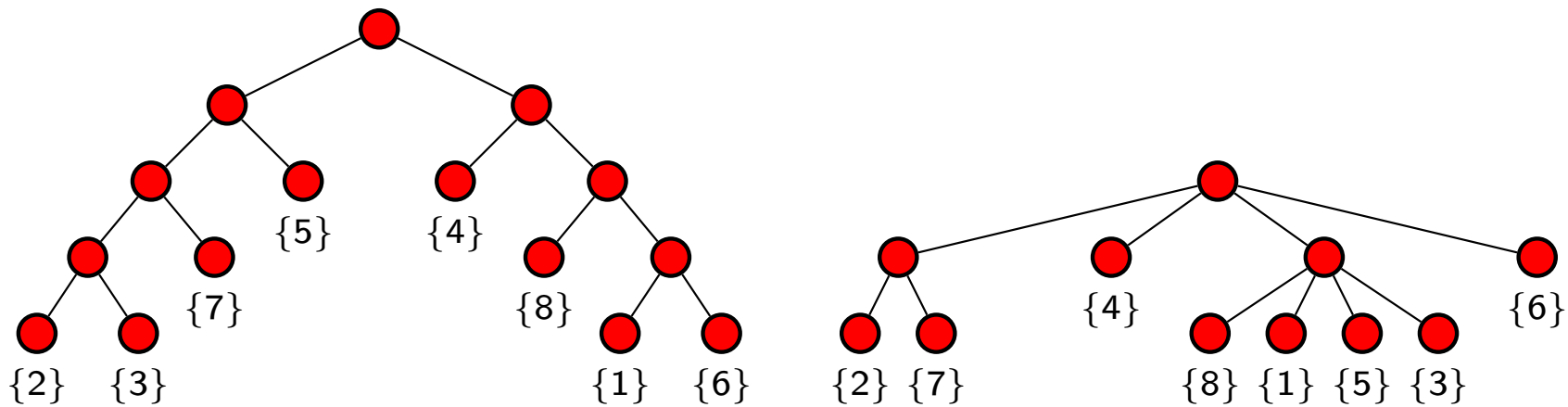
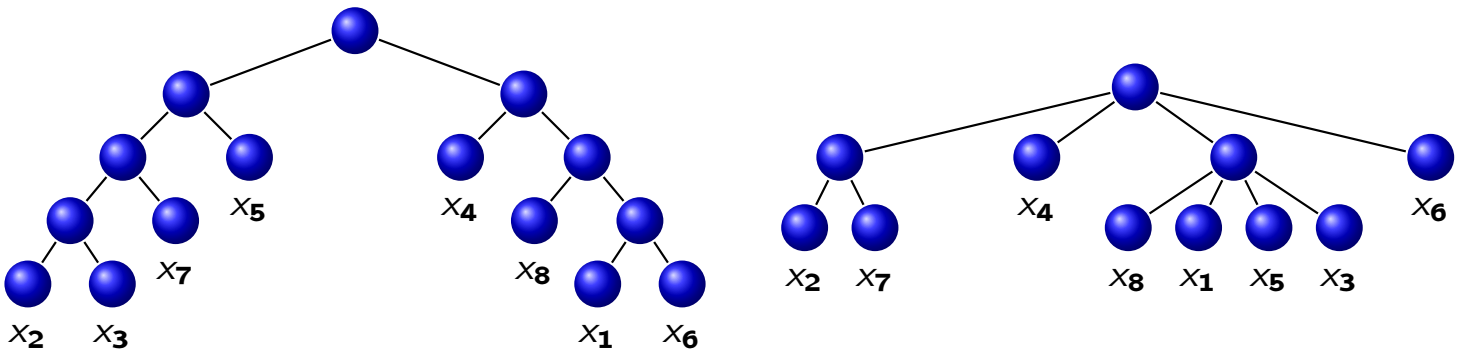
has complexity linear in d for some tree, exponential in d for other trees.
- How to select a good tree ? Combinatorial problem. Possible stochastic algorithms [Grelier et al 2018, Michel and Nouy 2022].



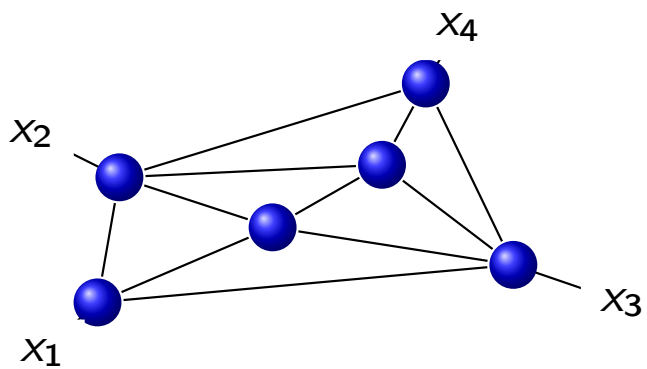
Some open questions

- What are the properties of the approximation tool with free tree T over $\{1, \dots, (L+1)d\}$

$$\Phi_n = \{f \in \Phi_{L,T,r,m} : L \in \mathbb{N}_0,\ T \subset 2^{\{1,\dots,(L+1)d\}},\ r \in \mathbb{N}^{\#T},\ compl(f) \leq n\} \quad ?$$



- What about approximation classes of more general tensor networks ?



Some open questions

- Algorithms to practically compute approximations achieving a certain precision with almost optimal complexity, using available information on the function (model equations, point samples...)
- Computational complexity of (deterministic or randomized) algorithms based on point samples for functions from approximation classes of tensor networks ?
- Theory to practice gap ?

Outline

Approximation theory of tree tensor networks

Given a function f from a Banach space X, the best approximation error of f by an element of Φ_n is

$$E(f, \Phi_n)_X := \inf_{g \in \Phi_n} \|f - g\|_X$$

Fundamental questions are:

- does $E(f, \Phi_n)_X$ converge to 0 for any f ?
  (universality)
- does a best approximation exist ?
  (proximinality)
- how fast does it converge for functions from classical function classes ?
  (expressivity)
- what are the functions for which $E(f, \Phi_n)_X$ converges with some given rate ?
  (characterization of approximation classes)

Outline

Universality

First note that for any algebraic feature tensor space V, and any tree T,

$$\bigcup_r \mathcal{T}_r^T(V) = V.$$

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- Consider the first family of approximation tools with variable feature spaces V_N, $N \in \mathbb{N}^d$.

  If $\bigcup_N V_N$ is dense in X, then the tools are universal for functions in X.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, and for polynomial or splines spaces V_N.

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- Consider the second family of approximation tools using tensorization.

  If $\bigcup_L V_L$ is dense in X, then the tools are universal for functions in X.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, assuming that S contains the function one.

Proximinality

For any tree T, any T-rank r, and any finite dimensional tensor space V of X, $\mathcal{T}_r^T(V)$ is a closed set in V.

$\Phi_n^{\mathcal{F}}$ (full tensor networks) is a finite union of such sets, all contained in a single finite dimensional space V^*. Then $\Phi_n^{\mathcal{F}}$ is a closed set of a finite dimensional space V^* and is therefore proximinal in X.

However, $\Phi_n^{\mathcal{F}}$ (sparse tensor networks) is not closed.

Expressivity

Different ways to analyse the expressivity of tree tensor networks

- Exploit known results on other approximation tools and estimate the complexity to encode these tools using tree tensor networks.
- Directly encode a function using tree tensor networks (with controlled errors)
- Analyse the convergence of bilinear approximations

$$u(x_\alpha, x_{\alpha^c}) \approx \sum_{k=1}^{r_\alpha} u_k^\alpha(x_\alpha) u_k^{\alpha^c}(x_{\alpha^c})$$

or the approximability of partial evaluations $u(\cdot, x_{\alpha^c})$ by linear approximation spaces of dimension r_α.

Encoding polynomials and splines

Polynomials

The tensorization of a polynomial of degree p has all ranks bounded by $p+1$.

Trigonometric polynomials

The tensorization of the function $\cos(\omega x + \varphi)$ has all ranks equal to 2.

Then the tensorization of a trigonometric polynomial of degree p has all ranks bounded by $2p+1$.

Free knot splines

A spline φ of degree p over N b-adic intervals forming a partition of $[0,1)$ is such that

$$\operatorname{rank}_{\{1,\dots,\nu\}}(\varphi) \leq \begin{cases} p+N, & 1 \leq \nu < \ell. \\ p+1, & \ell \leq \nu \leq L. \end{cases}$$

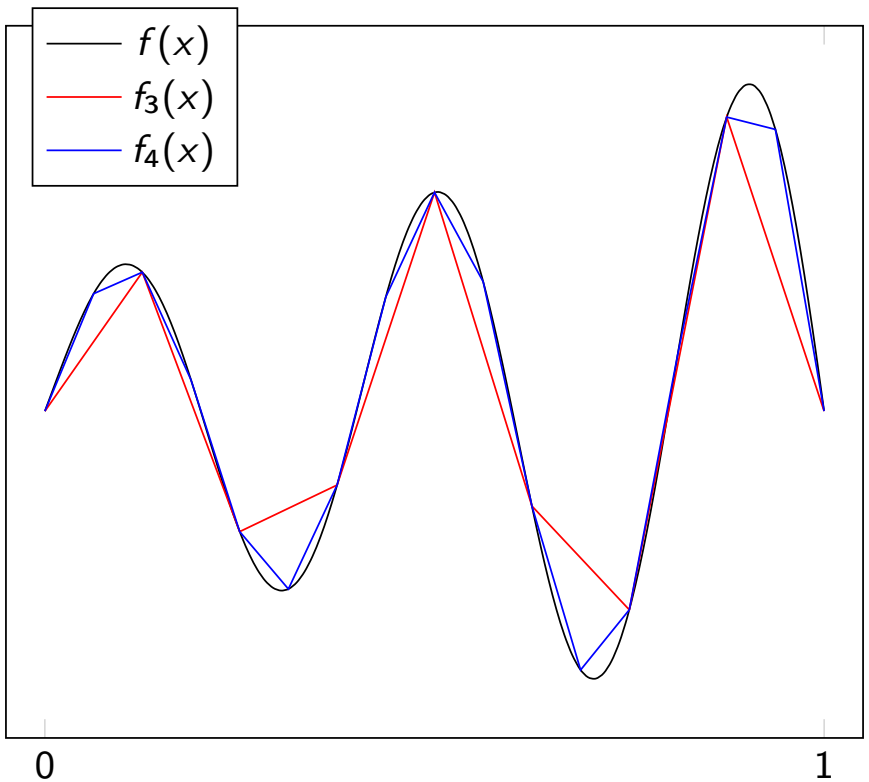
where $b^{-\ell}$ is the minimal length of intervals.

Encoding polynomials and splines

Ranks of interpolants

For a function f and its interpolation f_L onto V_L, the space of piecewise polynomials of degree m on a uniform partition of b^L intervals, it holds

$$\text{rank}_\alpha(\boldsymbol{f}_L) \leq \text{rank}_\alpha(\boldsymbol{f})$$



Encoding multi-resolution analysis

For a function $\psi : \mathbb{R} \to \mathbb{R}$ supported on $[0, 1]$, we define its level ℓ b-adic dilation, shifted by $j = 0, \ldots, b^L - 1$,

$$\psi_{\ell,j}(x) = \psi(b^\ell x - j)$$

Its tensorization at level ℓ is an elementary (rank-one) tensor

$$T_{b,\ell}\psi_{\ell,j} = e_{j_1} \otimes \ldots e_{j_\ell} \otimes \psi$$

with $j = [j_1, \ldots, j_\ell]_b$ and e_k the canonical basis vectors in $\mathbb{R}^b$.

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Its tensorization at level $L \geq l$ is

$$T_{b,L}\psi_{\ell,j} = e_{j_1} \otimes \ldots e_{j_\ell} \otimes (T_{b,L-\ell}\psi)$$

The (approximate) encoding of $\psi_{\ell,j}$ boils down to the (approximate) encoding of the mother function ψ with tensor networks.

In particular, if ψ is a (piecewise) polynomial, $\psi_{\ell,j}$ is encoded at precision ϵ using tensorization at level $L = \ell + O(\log(\epsilon^{-1}))$.

This yields a very efficient encoding of piecewise polynomial MRAs (B-spline wavelets).

Approximation of functions from Besov spaces $B_q^\alpha(L^p)$

From results on spline approximation and their encoding with tensor networks, we obtain

Theorem

Let $f \in B_q^\alpha(L^p)$ with $\alpha > 0$ and $0 < p, q \leq \infty$. Then

$$E(f, \Phi_n^{\mathcal{F}})_{L^p} \leq C n^{-\alpha/d} |f|_{B_\infty^\alpha(L^p)}$$

- Tensor networks achieve optimal rates for any Besov regularity order (measured in L^p norm).
- They perform as well as optimal linear approximation tools (e.g. splines), without requiring to adapt the tool to the regularity order α.
- The depth (resolution L) of the network is crucial to capture extra regularity ($\alpha > m + 1$).

Approximation of functions from Besov spaces $B_q^\alpha(L^\tau)$

Now consider the harder problem of approximating functions from Besov spaces $B_q^\alpha(L^\tau)$ where regularity is measured in a L^τ-norm weaker than L^p-norm.

From results on best n-term approximation using dilated splines, we obtain

Theorem

Let $f \in B_q^\alpha(L^\tau)$ with $\alpha > 0$, $0 < q \leq \tau < p < \infty$, $1 \leq p < \infty$ and

$$\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}.$$

Then

$$E(f, \Phi_n^{\mathcal{S}})_{L^p} \leq C n^{-\tilde{\alpha}/d} |f|_{B_q^\alpha(L^\tau)}, \quad E(f, \Phi_n^{\mathcal{F}})_{L^p} \leq C n^{-\tilde{\alpha}/(2d)} |f|_{B_q^\alpha(L^\tau)},$$

for arbitrary $\tilde{\alpha} < \alpha$.

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for arbitrary $\tilde{\alpha} < \alpha$.

- Sparse tensor networks achieve arbitrarily close to optimal rates in $O(n^{-\alpha/d})$ for functions with any Besov smoothness α (measured in L^τ norm), without the need to adapt the tool to the regularity order α.
- Here depth and sparsity are crucial for obtaining near to optimal performance.
- Full tensor networks have slightly lower performance in $O(n^{-\alpha/(2d)})$.

High-dimensional approximation

- For Besov spaces $B_q^\alpha(L^p)$, tensor networks achieve (near to) optimal rate in $O(n^{-\alpha/d})$ which deteriorates with d, that is the curse of dimensionality.

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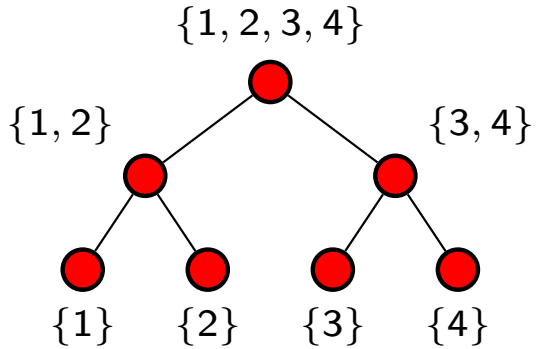
- Curse of dimensionality can be circumvented for non usual function classes such as compositions of smooth functions

Compositional functions

Consider a tree-structured composition of smooth functions $\{f_\alpha : \alpha \in T\}$, see [Mhaskar, Liao and Poggio 2016] for deep neural networks, and [Bachmayr, Nouy and Schneider 2023] for tree tensor networks.

$$f_{1,2,3,4}\left(f_{1,2}\left(f_1(x_1), f_2(x_2)\right), f_{3,4}\left(f_3(x_3), f_4(x_4)\right)\right)$$



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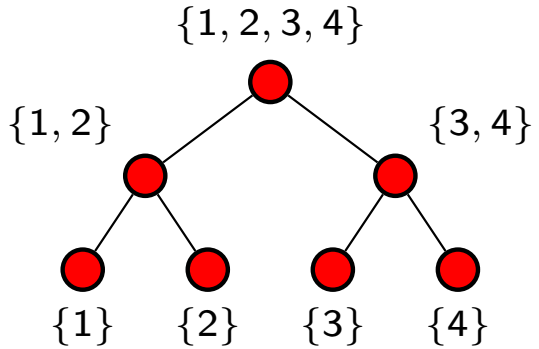

Assuming that the functions $f_\alpha \in W^{k,\infty}$ with $\|f_\alpha\|_{L^\infty} \leq 1$ and $\|f_\alpha\|_{W^{1,\infty}} \leq B$, the complexity to achieve an accuracy ϵ

$$n(\epsilon) \lesssim \epsilon^{-3/k}(L+1)^3 B^{3L} d^{1+3/2k}$$

with $L = \log_2(d)$ for a balanced tree and $L + 1 = d$ for a linear tree.

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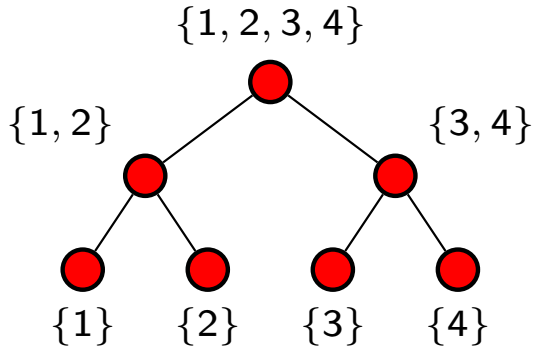

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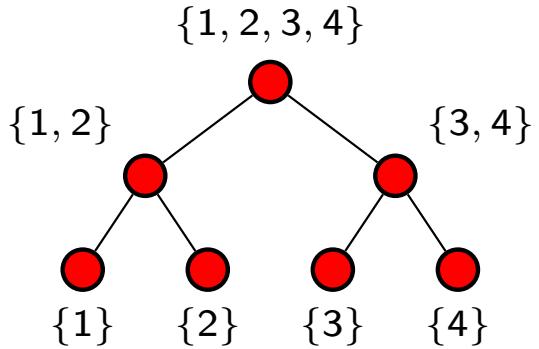

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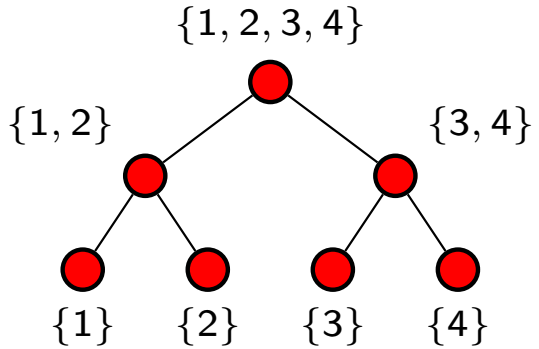

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- For $B \leq 1$, the complexity only scales polynomially in d whatever the tree: no curse of dimensionality !

More regularity, analytic functions

For function $f : [0, 1]$ with analytic extension on an open complex domain

$$D_\rho = \{z \in \mathbb{C} : dist(z, [0, 1]) < \frac{\rho - 1}{2}\}, \quad \rho > 1,$$

we obtain an exponential convergence

$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \leq C\gamma^{-n^{1/3}},$$

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The proof relies on the approximation of analytic functions with polynomials and the encoding of polynomials with tree tensor networks: a chebychev polynomial p of deree $\bar{m}$ is such that

$$\|f - p\|_{L^\infty} \leq \frac{2}{\rho-1}\|f\|_{L^\infty(D_\rho)}\rho^{-\bar{m}}$$

A polynomial of degree $\bar{m}$ can be approximated by φ in $\Phi_{L,r,m}$ with an error in $O(b^{-L(m+1)})$, so that

$$\|f - \varphi\|_{L^\infty} \lesssim \rho^{-\bar{m}} + b^{-L(m+1)}$$

We obtain the result by choosing $\bar{m} \sim n^{1/3}$ and $L \sim b^{-1}n^{1/3}$, so that $compl_{\mathcal{F}}(\varphi) \leq n$.

Analytic functions with singularities

Consider the approximation of $u(x) = x^\alpha$, $0 < \alpha \leq 1$, in L^∞.

- Piecewise constant linear approximation.

$$u \in B^\alpha_\infty(L^\infty), \quad u \notin B^\beta_\infty(L^\infty) \quad \text{for } \beta > \alpha,$$

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- Piecewise constant approximation and tensor networks.
  A piecewise constant approximation on a uniform mesh with 2^L elements exploiting low-rank structures gives an exponential convergence

$$E(f, \Phi^{\mathcal{F}}_n) \leq C\beta^{-n^\gamma}$$

  Achieves almost the performance of *h-p* methods [Kazeev and Schwab].

Beyond smoothness

Consider the Weierstrass function, continuous but nowhere differentiable

$$f(x) = \sum_{k=0}^{\infty} a^{-\alpha k} \cos(a^k \pi x), \quad a > 0, \quad 0 < \alpha \leq 1,$$

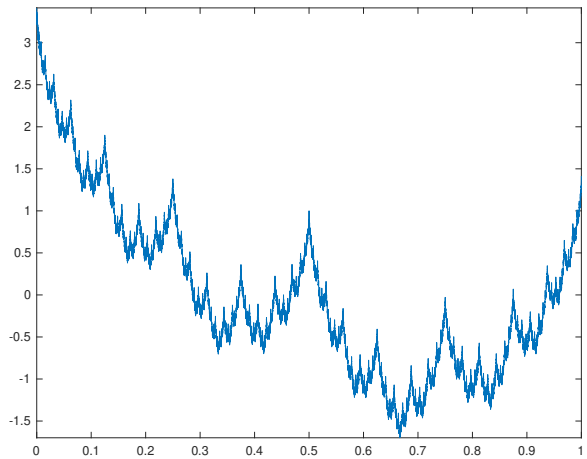


Figure: Weierstrass function for $\alpha = 1/2$, $a = 2$

We have an exponential convergence in L^∞-norm

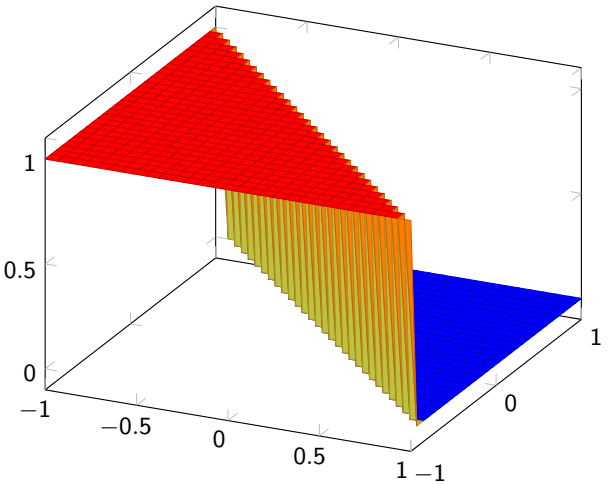
$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \lesssim \beta^{-n^{1/3}}$$

An error ϵ is achieved with resolution $L \sim \log(\epsilon^{-1})$, ranks $\sim \log(\epsilon^{-1})$ and complexity $n \sim \log(\epsilon^{-1})^3$

Discontinuous functions: the power of tensorization

Consider the problem of approximating the bivariate function on $(-1,1)^2$

$$u(x,t) = \begin{cases} 1 & \text{if } x+t<0 \\ 0 & \text{if } x+t \geq 0 \end{cases}$$



The manifold $K = \{u(\cdot,t) : t \in (-1,1)\}$ contains the indicator functions $1_{[-1,x_i]}(x)$, $x_i = -1+2i/m$. Therefore the balanced convex hull of K contains the orthogonal system $S = \{\psi_i(x) = \frac{1}{2}1_{(x_i,x_{i+1}]}(x) : 1 \leq i \leq m\}$ with $\|\psi_i\|_{L^2} = (2m)^{-1/2}$ and by taking $m = 2n$, we deduce

$$d_n(K)_{L^2} \geq 1/(2\sqrt{2})n^{-1/2},$$

so that the best rank-n approximation

$$u_n(x,t) = \sum_{i=1}^{n} v_i(x)w_i(t)$$

does not converge better than $\|u-u_n\|_{L^2} \gtrsim n^{-1/2}$.

Discontinuous functions: the power of tensorization

A piecewise constant interpolant u^L on a uniform grid with mesh size 2^{-L} is such that

$$\|u - u^L\|_{L^2} \leq meas(\{(x,t) : u \neq u^L\})^{1/2} \leq 2^{1/2} 2^{-L/2}$$

Using a tensorization $\tilde{\boldsymbol{u}}^L(i_1^x, ..., i_L^x, i_1^t, ..., i_L^t)$, we have

$$rank_{\{1,...,L\}}(\tilde{\boldsymbol{u}}^L) = \text{rank}\, u_L \sim 2^L$$

that means an encoding complexity in tensor train format $compl(\tilde{\boldsymbol{u}}^L) \gtrsim 2^{2L}$, which yields an approximation error $\gtrsim n^{-1/4}$.

However, the tensorization $\boldsymbol{u}^L(i_1^x, i_1^t, ..., i_L^x, i_L^t)$ of $u^L(x,t)$ satisfies

$$rank_{\{1,...,\nu\}}(\boldsymbol{u}^L) \leq 3$$

for all ν. Therefore, using tensor train format, $compl(u^L) \leq 36L$ and

$$E(u, \Phi_n^{\mathcal{F}})_{L^2} \leq 2^{1/2} 2^{-n/72}.$$

Outline

Canonical versus tree-based format

Consider a finite dimensional tensor space $V = V^1 \otimes \ldots \otimes V^d$ with $\dim(V_\nu) = \mathbb{R}^N$, which is identified with $\mathbb{R}^{N \times \ldots \times N}$. Denote by $\mathcal{R}_r = \{v : \text{rank}(v) \leq r\}$ and $\mathcal{T}_r^T = \{v : \text{rank}_\alpha(v) \leq r, \alpha \in T\}$.

- From canonical format to tree-based format.
  For any v in V and any $\alpha \subset D$, the α-rank is bounded by the canonical rank:

$$\text{rank}_\alpha(v) \leq \text{rank}(v).$$

  Therefore, for any tree T,

$$\mathcal{R}_r \subset \mathcal{T}_r^T,$$

  so that an element in $\mathcal{R}_r$ with storage complexity $O(dNr)$ admits a representation in $\mathcal{T}_r^T$ with a storage complexity $O(dNr + dr^{s+1})$ where s is the arity of the tree T.

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- From tree-based format to canonical format. For a balanced or linear binary tree, the subset

$$S = \{v \in \mathcal{T}_r^T : \text{rank}(v) < q^{d/2}\}, \quad q = \min\{N, r\},$$

  is of Lebesgue measure 0.

  Then a typical element $v \in \mathcal{T}_r^T$ with storage complexity of order $dNr + dr^3$ admits a representation in canonical format with a storage complexity of order $dNq^{d/2}$.

Influence of the tree

- For some functions, the choice of tree is not crucial. For example, an additive function

$$u_1(x_1) + \ldots + u_d(x_d)$$

has α-ranks equal to 2 whatever $\alpha \subset D$.

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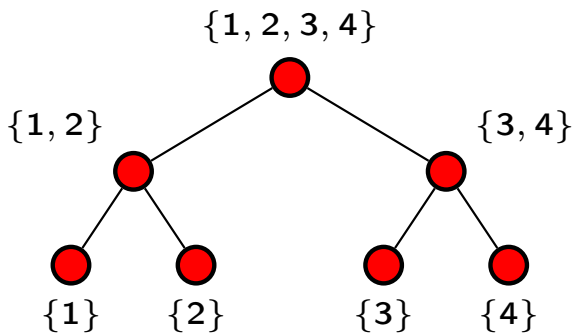
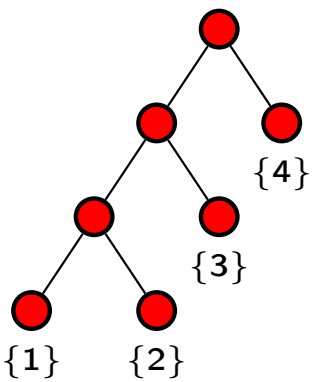
- But usually, different trees lead to different complexities of representations.



T^B (Balanced tree)



T^L (Linear tree)

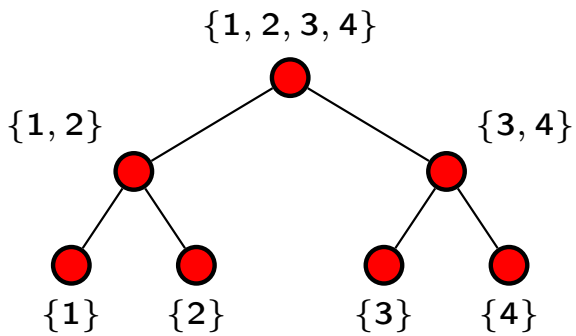
- If $\text{rank}_{T^L}(u) \leq r$ then $\text{rank}_{T^B}(u) \leq r^2$
- If $\text{rank}_{T^B}(u) \leq r$ then $\text{rank}_{T^L}(u) \leq r^{\log_2(d)/2}$

Influence of the tree

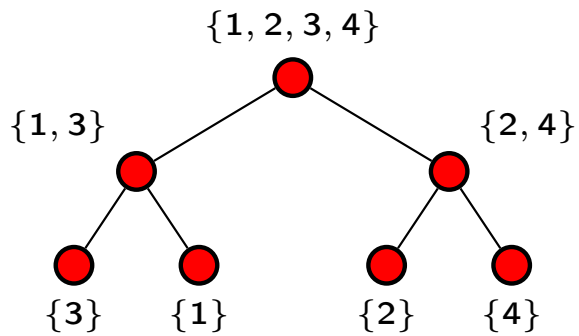
Given a tree T and a permutation σ of $D = \{1, \ldots, d\}$, we define a tree T_σ

$$T_\sigma = \{\sigma(\alpha) : \alpha \in T\}$$

having the same structure as T but different nodes.



T



T_σ with $\sigma = (3, 1, 2, 4)$

If $\text{rank}_T(u) \leq r$ then $\text{rank}_{T_\sigma}(u)$ typically depends on d.

Influence of the tree

- Consider the Henon-Heiles potential

$$u(x) = \frac{1}{2}\sum_{i=1}^{d} x_i^2 + 0.2\sum_{i=1}^{d-1}(x_i x_{i+1}^2 - x_i^3) + \frac{0.2^2}{16}\sum_{i=1}^{d-1}(x_i^2 + x_{i+1}^2)^2$$

Using a linear tree $T = \{\{1\}, \{2\}, \ldots, \{d\}, \{1,2\}, \{1,2,3\}, \ldots, \{1,\ldots,d-1\}, D\}$,

$$\mathrm{rank}_T(u) \leq 4, \quad \mathit{storage}(u) = O(d)$$

but for the permutation

$$\sigma = (1, 3, \ldots, d-1, 2, 4, \ldots, d) \qquad (\star)$$

and the corresponding linear tree T_σ,

$$\mathrm{rank}_{T_\sigma}(u) \leq 2d+1, \quad \mathit{storage}(u) = O(d^3).$$

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- For a typical tensor in $\mathcal{T}_r^T$ with T a binary tree, its representation in tree based format with tree T_σ, with σ as in $(\star)$, has a complexity scaling exponentially with d.
- As an example, consider the function $u(x,t) = 1_{x+t<0}$identified (through tensorization) with tensors $\boldsymbol{u}(i_1^x, \ldots, i_L^x, y^x, i_1^t, \ldots, i_L^t, y^t)$ and $\boldsymbol{u}(i_1^x, i_1^y, \ldots, i_L^x, i_L^y, y^x, y^t)$. Huge impact of the ordering !

Influence of the tree

- Consider the probability distribution $f(x) = \mathbb{P}(X = x)$ of a Markov chain $X = (X_1, \ldots, X_d)$ given by

$$f(x) = f_1(x_1) f_{2|1}(x_2|x_1) \ldots f_{d|d-1}(x_d|x_{d-1})$$

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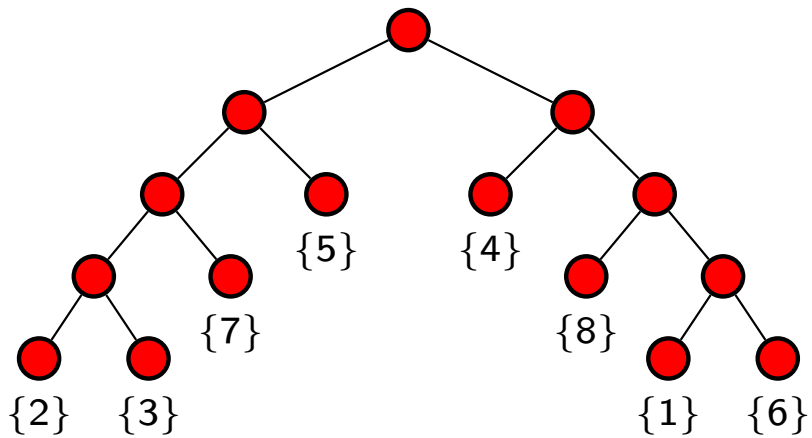
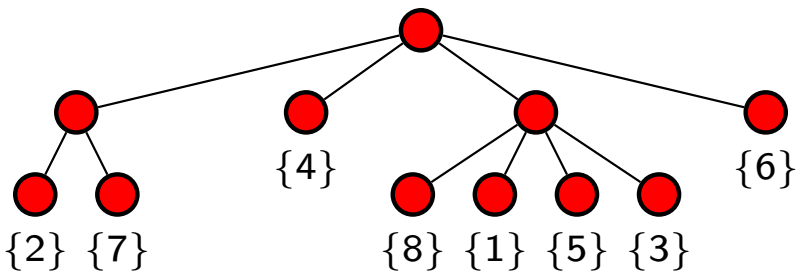
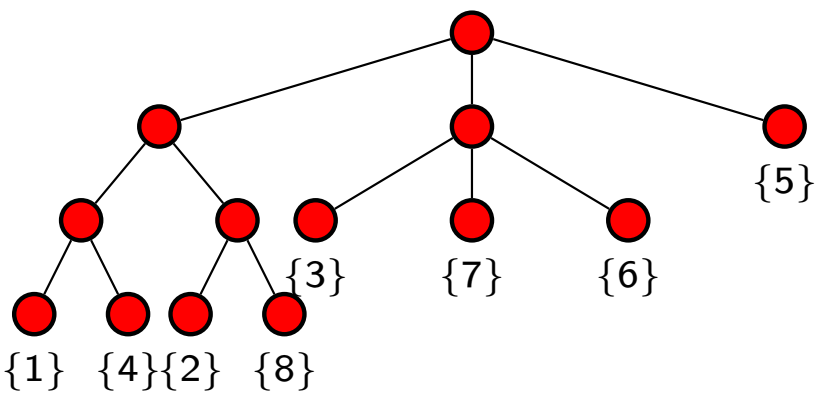
  - With the linear tree T containing interior nodes $\{1,2\}, \{1,2,3\}, \ldots, \{1, \ldots, d-1\}$, f admits a representation in tree-based format with storage complexity in dr^4.
  - The canonical rank of f is exponential in d.
  - But when considering the linear tree T_σ obtained by applying permutation $\sigma = (1, 3, \ldots, d-1, 2, 4, \ldots, d)$ to the tree T, the storage complexity in tree-based format is also exponential in d.

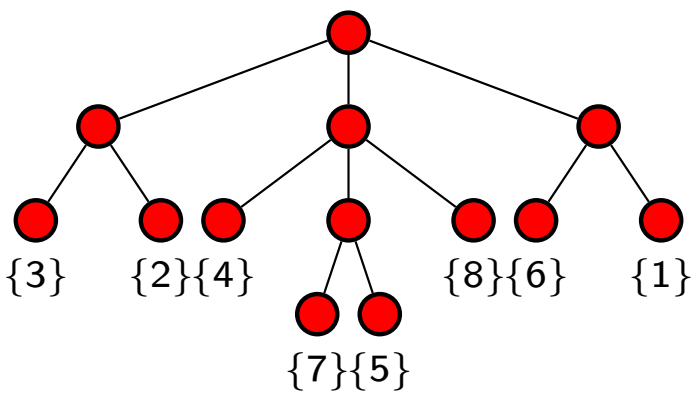
How to choose a good tree ?

A combinatorial problem...









Outline

Properties of tree tensor networks

We here consider approximation tools $(\Phi_n)_{n\geq 1}$ based on tensorization and tensor train format (with or without sparsity).
They satisfy

(P1) $\Phi_0 = \{0\}$, $0 \in \Phi_n$

(P2) $a\Phi_n = \Phi_n$ for any $a \in \mathbb{R} \setminus \{0\}$ (cone)

(P3) $\Phi_n \subset \Phi_{n+1}$ (nestedness)

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(P4) $\Phi_n + \Phi_n \subset \Phi_{cn}$ for some constant c (not too nonlinear)

For $X = L^p$, they further satisfy

(P5) $\bigcup_n \Phi_n$ is dense in L^p for $0 < p < \infty$ (universality),

(P6) for each $f \in L^p$ for $0 < p \leq \infty$, there exists a best approximation in $\Phi_n^{\mathcal{F}}$ (proximinal sets). However, $\Phi_n^{\mathcal{S}}$ is not closed.

Approximation classes

For an approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$, we define for any $\alpha > 0$ the approximation class

$$A^\alpha_\infty(L^p) := A^\alpha_\infty(L^p, \Phi)$$

of functions $f \in L^p$ such that

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- Full and sparse complexity measures yield two different approximation spaces

$$\mathcal{F}^\alpha_\infty(L^p) = A^\alpha_\infty(L^p, \Phi^{\mathcal{F}}), \quad \mathcal{S}^\alpha_\infty(L^p) = A^\alpha_\infty(L^p, \Phi^{\mathcal{S}})$$

such that

$$\mathcal{F}^\alpha_\infty(L^p) \hookrightarrow \mathcal{S}^\alpha_\infty(L^p) \hookrightarrow \mathcal{F}^{\alpha/2}_\infty(L^p)$$

Direct embeddings

From results on the approximation properties for Besov spaces, we have the following results.

- (Linear approximation) For $\alpha > 0$ and $0 < p \leq \infty$,

$$B_q^\alpha(L^p) \hookrightarrow \mathcal{F}_\infty^{\alpha/d}(L^p),$$

$$MB_q^\alpha(L^p) \hookrightarrow \mathcal{S}_\infty^\alpha(L^p),$$

$$AB_q^{\boldsymbol{\alpha}}(L^p) \hookrightarrow \mathcal{S}_\infty^{s/d}(L^p)$$

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- (Nonlinear approximation) For $\alpha > 0$, $1 \leq p < \infty$, $0 < q \leq \tau < p < \infty$ and $\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}$,

$$B_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p),$$

$$MB_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/2}(L^p)$$

  for arbitrary $\tilde{\alpha} < \alpha$, and

$$AB_q^{\boldsymbol{\alpha}}(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p)$$

  for arbitrary $\tilde{\alpha} < s(\boldsymbol{\alpha})$.

Interpolation family

The properties of Φ_n allow to apply classical results from approximation theory, in particular to deduce from embedding results on $A^\alpha_\infty(L^p)$ embedding results on interpolation spaces

$$A^\beta_q(L^p) = (L^p, A^\alpha_\infty(L^p))_{\beta/\alpha,q}, \quad 0 < \beta < \alpha, \qquad 0 < q \leq \infty$$

that are quasi-Banach spaces with quasi-norm

$$\|f\|_{A^\alpha_q} = \|f\|_{L^p} + |f|_{A^\alpha_q}, \quad |f|_{A^\alpha_q} = \left(\sum_{n=1}^{\infty} n^{-1} \left(n^\alpha E(f, \Phi_n)_X \right)^q \right)^{1/q}$$

(functions with faster convergence than those of $A^\alpha_\infty(L^p)$).

No inverse embedding

For any $\alpha > 0$, $q \leq \infty$, and any β,

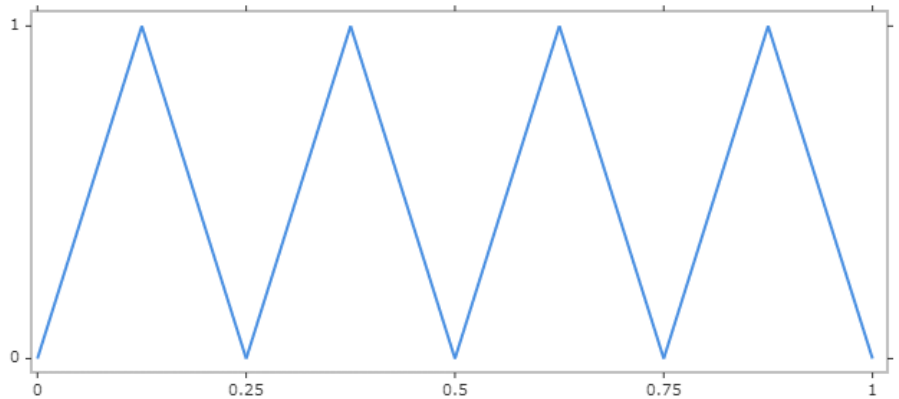
$$\mathcal{F}_q^\alpha(L^p) \not\hookrightarrow B_q^\beta(L^p).$$

That means that approximation classes contain functions that have no smoothness in a classical sense.

Tree tensor networks may be useful for the approximation of functions beyond standard smoothness classes.

No inverse embedding

This is proved by contradiction by considering the sawtooth function φ_L with 2^L teeth such that $\varphi_L \in \Phi_n$ with $n \sim L$.



From properties (P1)-(P6), $\mathcal{F}_q^\alpha(L^p)$ satisfies the Berstein inequality, that is

$$\|\varphi\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha \|\varphi\|_{L^p} \quad \forall \varphi \in \Phi_n.$$

Moreover, $\|\varphi_L\|_{L^p} \sim 1$ and $\|\varphi_L\|_{B_q^\beta(L^p)} \gtrsim 2^{\beta L}$. If the embedding were true, we would have

$$2^{\beta n} \lesssim \|\varphi_L\|_{B_q^\beta(L^p)} \lesssim \|\varphi_L\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha,$$

a contradiction.

The role of depth

Consider the approximation with restricted resolution

$$\Phi_n^{\mathcal{L}} = \{f \in \Phi_n : L(f) \leq \mathcal{L}(n)\}$$

where $L(f)$ is the minimal resolution L such that $f \in V_L$, and $\mathcal{L}$ some growth function.

Since $L(f) \leq n$ for $f \in \Phi_n$, $\Phi_n^{\mathcal{L}} = \Phi_n$ for $\mathcal{L} = n$.

In dimension $d = 1$, for $\mathcal{L}(n) = r \log_b(n) + c$, the following Bernstein inequality holds

$$|f|_{B_\tau^{m+\mathbf{1}}(L^\tau)} \lesssim \|f\|_{L^p} b^{c(m+\mathbf{1})} n^{r(m+\mathbf{1})}$$

with τ the Sobolev embedding number, and m the local polynomial degree. This implies the inverse embedding of the corresponding approximation class

$$A_\infty^\alpha(L^p; (\Phi_n^{\mathcal{L}})) \hookrightarrow B_\tau^{\alpha/(m+\mathbf{1})}(L^\tau)$$

Hence the importance of depth L for going beyond standard regularity classes.

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