Low-rank tensor methods for parametric and stochastic problems

Part 2: Sparse and low-rank tensor methods for high-dimensional approximation

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What we learned from Part 1

$$u \in X = L^p_\mu(\Xi; \mathcal{V})$$

- Subspace-based model reduction related with low-rank approximation

$$u(\xi) \approx \sum_{i=1}^{n} v_i s_i(\xi) \in \mathcal{V}_n \otimes L^p_\mu(\Xi)$$

- Optimal reduced spaces $\mathcal{V}_n = span\{v_1, \ldots, v_n\}$ can be defined w.r.t. L^p_μ-norm
- Algorithms for quasi-optimal constructions: greedy or not greedy.

Remaining issue: high-dimensional approximation

$$u(\xi) = u(\xi_1, \ldots, \xi_d)$$

- Different types of approximations, usually linear approximation

$$u_\Lambda(\xi) \in X_\Lambda = \left\{ \sum_{\alpha \in \Lambda} u_\alpha \psi_\alpha(\xi) : u_\alpha \in \mathcal{V} \right\}$$

with classical bases: polynomials, piecewise polynomials, wavelets...

- Different types of constructions depending on the expected accuracy:
  - uniform accuracy (for optimization, quantile estimation, ...) :

$$\|u(\xi) - u_\Lambda(\xi)\| \leq \epsilon \quad \forall \xi \in \Xi$$

  - mean-squared accuracy (for statistical moments, global sensitivity...) :

$$\int_\Xi \|u(\xi) - u_\Lambda(\xi)\|^2 \mu(d\xi) \leq \epsilon$$

  - ...

- Different types of constructions depending on the available information:
  - point evaluations $\Rightarrow$ interpolation, approximate projection using quadrature, discrete projection
  - model equations $\Rightarrow$ Galerkin projection
- Quasi-optimality under continuity and stability properties

$$\|u - u_\Lambda\|_X \leq C \inf_{v \in X_\Lambda} \|u - v\|_X$$

- The analysis of best approximation error $\inf_{v \in X_\Lambda} \|u - v\|$ requires extra-information on approximation spaces and the solution (regularity w.r.t. ξ)

Outline

1. Polynomial approximation spaces
2. High-dimensional approximation - tractability
3. Best n-term approximation and quasi-optimal approximation spaces
4. Higher-order tensors and low-rank tensor formats
5. Approximation in low-rank tensor formats
6. Algorithms for computing low-rank approximations
7. Higher-order low-rank methods for parametric and stochastic equations

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Notations, definitions

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we use the following notations.

- For $1 \leq p < \infty$,

$$|\alpha|_p = \left(\sum_{k=1}^{d} \alpha_k^p \right)^{1/p},$$

  and

$$|\alpha|_\infty = \max_{1 \leq k \leq d} \alpha_k$$

- $|\alpha| = |\alpha|_1$
- $\alpha! = \prod_{k=1}^d \alpha_k!$
- For $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$, $\alpha^\omega = \prod_{k=1}^d \alpha_k^{\omega_k}$
- For $u : \Xi \to X$,

$$D^\alpha u(\xi) = \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} u(\xi_1, \dots, \xi_d) = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \dots \partial \xi_d^{\alpha_d}} u(\xi_1, \dots, \xi_d)$$

Polynomial spaces

- For a set of multi indices $\Lambda \subset \mathbb{N}^d$, we define the polynomial space

$$\mathbb{P}_\Lambda(\Xi) = \mathrm{span}\left\{ \xi^\alpha = \prod_{k=1}^{d} \xi_k^{\alpha_k} : \alpha \in \Lambda \right\}$$

- Orthonormal basis $\{\psi_\alpha\}_{\alpha \in \Lambda}$ of $\mathbb{P}_\Lambda(\Xi) \subset L^2_\mu(\Xi)$ such that

$$\mathbb{P}_\Lambda(\Xi) = \mathrm{span}\left\{ \psi_\alpha(\xi) : \alpha \in \Lambda \right\}$$

- If product measure $\mu = \mu_1 \otimes \ldots \otimes \mu_d$ on $\Xi_1 \times \ldots \times \Xi_d$, then

$$\psi_\alpha(\xi) = \psi^{(1)}_{\alpha_1}(\xi_1) \ldots \psi^{(d)}_{\alpha_d}(\xi_d)$$

  with orthonormal polynomials $\{\psi^{(k)}_j\}_{j \geq 0}$ in $L^2_{\mu_k}(\Xi_k)$.

Polynomial approximation spaces

- Full tensor product (bounded partial degree)

$$\Lambda = \{\alpha : \alpha_k \leq p_k, 1 \leq k \leq d\}$$

$$\#\Lambda = \prod_{k=1}^{d} (p_k + 1)$$

$$\mathbb{P}_\Lambda(\Xi) = \mathbb{P}_{p_1}(\Xi_1) \otimes \ldots \otimes \mathbb{P}_{p_d}(\Xi_d)$$

Convergence of polynomial approximations

- For $X_{\Lambda_p} = \mathcal{V} \otimes \mathbb{P}_{\Lambda_p}(\Xi)$, let

$$\|u - u_{\Lambda_p}\|_{L^2_\mu(\Xi;\mathcal{V})} = \inf_{v \in X_{\Lambda_p}} \|u - v\|_{L^2_\mu(\Xi;\mathcal{V})}$$

- We have

$$\|u - u_{\Lambda_p}\|_{L^2_\mu(\Xi;\mathcal{V})} \leq \inf_{v \in X_{\Lambda_p}} \|u - v\|_{L^\infty_\mu(\Xi;\mathcal{V})} := \epsilon_p^{(\infty)}$$

so that results in L^2 norm can be deduced from stronger results in L^∞ norm.

Convergence of polynomial approximations

The following result can be found in [Chen-Quarteroni-Rozza 2014].

- Consider $\Xi = (-1,1)^d$ and $\Lambda_p = \{\alpha : \alpha_k \leq p_k\}$ and

$$\mathbb{P}_{\Lambda_p}(\Xi) = \otimes_{k=1}^d \mathbb{P}_{p_k}(\Xi_k) \quad \text{(full tensor product space)}$$

- Assume $u : \Xi \to \mathcal{V}$ is analytic and can be analytically extended to $\widehat{\Xi} = \left\{z \in \mathbb{C}^d : dist(z_k, \Xi_k) \leq \tau_k, 1 \leq k \leq d\right\}$. Then

$$\boxed{\epsilon_p^{(\infty)} \leq \sum_{k=1}^d C_{p_k} r_k^{-p_k}}$$

  where $r_k = \sqrt{1+\tau_k^2} + \tau_k > 1$ and $C_{p_k} \leq C \log(p_k + 1)$.

- In the case $p_k = p_\star$ for all k, then with $r_\star = \min_k r_k$,

$$\epsilon_p^{(\infty)} \leq C_{p_\star} d r_\star^{-p_\star}$$

  or with $n = \#\Lambda_p = (p_\star + 1)^d$,

$$\boxed{\epsilon_p^{(\infty)} \leq C_n r_\star^{1-n^{1/d}}}$$

  where $C_n \leq C \log(n)$.

- Convergence deteriorates with the dimension d: curse of dimensionality

Piecewise polynomial approximations

See e.g. [Deb-Babuska 2001, Babuska-Tempone-Zouraris 2004, Wan-Karniadakis 2005, Frauenfelder-Schwab-Todor 2005].

- For a given partition $(K^t)_{t\in T}$ of Ξ, let

$$\mathcal{S}_{\Xi,\Lambda} = \left\{ v : \Xi \to \mathbb{R} : v_{|K^t} \in \mathbb{P}_{\Lambda_t}(K^t) \text{ for all } t \in T \right\}$$

- As a standard case, assume $\Xi = \times_k \Xi_k$ with Ξ_k a bounded interval. Introduce a partition made of boxes $K^t = \times_{k=1}^d K_k^t$ with $K_k^t = (a_k^t, b_k^t) \subset \Xi_k$ with maximal sizes $h_k = \max_t |b_k^t - a_k^t|$ in dimension k. Then introduce

$$\mathcal{S}_{h,p} = \left\{ v : \Xi \to \mathbb{R} : v_{|K^t} \in \mathbb{P}_p(K^t) = \otimes_{k=1}^d \mathbb{P}_{p_k}(K_k^t) \right\}$$

Piecewise polynomial approximations

- If $u \in C^{p+1}(\Xi; \mathcal{V})$, it holds

$$\epsilon_{h,p}^{(2)} := \inf_{v \in \mathcal{V} \otimes \mathcal{S}_{h,p}} \|u - v\|_{L^2_\mu(\Xi;\mathcal{V})} \leq C \sum_{k=1}^{d} \left(\frac{h_k}{2}\right)^{p_k+1} \frac{\|\partial_{\xi_k}^{p_k+1} u\|_{L^2_\mu(\Xi;\mathcal{V})}}{(p_k+1)!}$$

$$\leq \sum_{k=1}^{d} C_{p_k} h_k^{p_k+1} \|u\|_{C^{p+1}(\Xi;\mathcal{V})}$$

  with C_{p_k} depending on p_k.

- If $p_k = p_\star$ and $h_k = h$ for all k, then

$$\epsilon_{h,p}^{(2)} \leq C_{p_\star} d h^{p_\star+1} \|u\|_{C^{p_\star+1}(\Xi;\mathcal{V})}$$

  With $n = \dim(\mathcal{S}_{h,p}) \sim (h^{-1}(p_\star+1))^d$,

$$\epsilon_{h,p}^{(2)} \lesssim D_{p_\star} d n^{-\frac{p_\star+1}{d}}$$

- Convergence deteriorates with the dimension d: curse of dimensionality $\Rightarrow$ requires adaptivity.

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High-dimensional approximation

- Consider the approximation of the multivariate function

$$u(x_1, \dots, x_d)$$

- Tensorization of bases can be extended to other types of bases:

$$\psi_\alpha(x_1, \dots, x_d) = \psi^{(1)}_{\alpha_1}(x_1) \dots \psi^{(d)}_{\alpha_d}(x_d)$$

- Tensor product discretization yields high-dimensional parametrizations

$$u(x_1, \dots, x_d) \approx \sum_{\alpha_1=1}^{n} \dots \sum_{\alpha_d=1}^{n} a_{\alpha_1 \dots \alpha_d} \psi^{(1)}_{\alpha_1}(x_1) \dots \psi^{(d)}_{\alpha_d}(x_d), \quad a \in \mathbb{R}^{n^d}$$

  - Number of parameters for full tensor product approximation

$$N = n^d$$

  - Assuming $u \in C^s((0,1)^d)$, accuracy

$$\epsilon = O(n^{-s}) = O(N^{-s/d})$$

    and the number of parameters to achieve accuracy ϵ is

$$N = O(\epsilon^{-d/s}) \quad \text{(Curse of dimensionality)}$$

Curse of dimensionality - tractability (Novak,Wozniakowski)

- Curse of dimensionality related to computational intractability in high dimension.
  - Quantitive definition through information-based complexity analysis
  - Tractability depends on the measure of precision and the available information.

- $N(\epsilon, d)$ being the number of linear informations to obtain a precision ϵ, intractability (curse of dimensionality) when

$$\lim_{\epsilon^{-1}+d\to\infty} \frac{\log N(\epsilon, d)}{\epsilon^{-1}+d} > 0$$

Curse of dimensionality - tractability (Novak,Wozniakowski)

- Weak tractability

$$\lim_{\epsilon^{-1}+d\to\infty} \frac{\log N(\epsilon, d)}{\epsilon^{-1} + d} = 0$$

- Polynomial tractability

$$N(\epsilon, d) \leq C\epsilon^{-p} d^q \quad \text{for all } \epsilon \text{ and } d$$

- Strong polynomial tractability

$$N(\epsilon, d) \leq C\epsilon^{-p} \quad \text{for all } \epsilon$$

Intractability for the approximation of smooth functions

Consider the set of functions

$$F_d = \left\{ u \in C^\infty((0,1)^d) : \sup_\alpha \|D^\alpha u\|_\infty < \infty \right\}$$

- Minimal approximation error using linear informations:

$$\epsilon(N, d) = \inf_{A_N} \sup_{u \in F_d} \|u - A_N(u)\|_\infty$$

  where the infimum is taken over all algorithms A_N using N linear informations on u and providing an approximation $A_N(u)$.

- Optimal rate of convergence is infinite: for arbitrary large s

$$\epsilon(N, d) = O(N^{-s}) \quad \text{as } N \to \infty$$

$$N(\epsilon, d) = O(\epsilon^{-1/s}) \quad \text{as } \epsilon \to 0$$

- But what about the constants in O ? [Novak 2009] proves

$$N(\epsilon, d) \geq 2^{\lfloor \frac{d}{2} \rfloor}$$

Curse of dimensionality !

Remedies: structured approximations

Order reduction methods must exploit specific structures (application dependent)

- Anisotropic smoothness
- Low effective dimensionality, e.g.

$$u(x_1, \dots, x_d) \approx g(x_1, x_2)$$

- Low-order interactions, e.g.

$$u(x_1, \dots, x_d) \approx u_0 + \sum_i u_i(x_i) + \sum_{i \neq j} u_{i,j}(x_i, x_j)$$

- Sparsity (relatively to a basis or frame)

$$u(x_1, \dots, x_d) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha \psi_\alpha(x_1, \dots, x_d) \approx \sum_{\alpha \in \Lambda} a_\alpha \psi_\alpha(x_1, \dots, x_d)$$

Structured (sparse) tensorization

$$\mathcal{S}_\Lambda = span\left\{\psi_\alpha(x) = \psi^{(1)}_{\alpha_1}(x_1)\dots\psi^{(d)}_{\alpha_d}(x_d) : \alpha \in \Lambda\right\}$$

- Total degree: isotropic (left), anisotropic (right)

$$\Lambda = \{\alpha : \sum_k \alpha_k \leq p\} \qquad \Lambda = \{\alpha : \sum_k \omega_k \alpha_k \leq p\}$$

- Hyperbolic Cross: isotropic (left), anisotropic (right)

$$\Lambda = \{\alpha : \prod_k (\alpha_k + 1) \leq p + 1\} \qquad \Lambda = \{\alpha : \prod_k (\alpha_k + 1)^{\omega_k} \leq p + 1\}$$

And now the question is ....

Can we define optimal or quasi-optimal approximation spaces ?

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Best n-term approximation

- Let $\{\psi_\alpha\}_{\alpha \in \mathbb{N}^d}$ be an orthonormal basis of $L^2_\mu(\Xi)$.
- Let $u \in X = L^2_\mu(\Xi; \mathcal{V})$ such that

$$u = \sum_\alpha u_\alpha \psi_\alpha, \quad u_\alpha = \mathbb{E}_\mu(u(\xi)\psi_\alpha(\xi))$$

- For a index set Λ, let

$$X_\Lambda = \left\{ u(\xi) = \sum_{\alpha \in \Lambda} u_\alpha \psi_\alpha(\xi) : u_\alpha \in \mathcal{V} \right\} \subset X$$

- The best approximation u_Λ of u in X_Λ is defined by

$$\|u - u_\Lambda\|_{L^2_\mu(\Xi;\mathcal{V})} = \min_{v \in X_\Lambda} \|u - v\|_{L^2_\mu(\Xi;\mathcal{V})}$$

  and such that

$$\|u - u_\Lambda\|^2_{L^2_\mu(\Xi;\mathcal{V})} = \| \sum_{\alpha \notin \Lambda} u_\alpha \psi_\alpha \|^2_{L^2_\mu(\Xi;\mathcal{V})} = \sum_{\alpha \notin \Lambda} \|u_\alpha\|^2_{\mathcal{V}}$$

Best n-term approximation

- Best n-term approximation

$$\sigma_n^{(2)} = \min_{\#\Lambda_n = n} \min_{v \in X_{\Lambda_n}} \|u - v\|_{L^2_\mu(\Xi;\mathcal{V})}$$

where the minimum is taken over all subsets Λ_n with cardinal n. Optimal Λ_n obtained by retaining the n largest coefficients $\|u_\alpha\|_{\mathcal{V}}$.

- First question: How fast $\sigma_n^{(2)}$ converges with n ?
- Second question : How to construct Λ_n in practice (since u_α are not available) and how $\|u - u_{\Lambda_n}\|_X$ does it compare with $\sigma_n^{(2)}$?

Convergence of best n-term approximation: some results

- If the sequence $(\|u_\alpha\|_{\mathcal{V}})_\alpha \in \ell^p$ with $p < 1$, then there exists Λ_n with $\#\Lambda_n = n$ such that
$$\|u - u_{\Lambda_n}\|_{L^2_\mu(\Xi;\mathcal{V})} \leq C n^{-s}, \quad C = \|(\|u_\alpha\|_{\mathcal{V}})_\alpha\|_{\ell^p}, \quad s = \frac{1}{p} - \frac{1}{2}$$
so that
$$\sigma_n^{(2)} \leq C n^{-s}$$
- See [Cohen-DeVore-Schwab 2010,Chkifa-Cohen-Schwab 2014] for a proof of $(\|u_\alpha\|_{\mathcal{V}})_\alpha \in \ell^p$ for a large class of parametric problems. Results are working for infinitely many random variables.

Remark on L^∞ case

- For $u \in L^\infty(\Xi; \mathcal{V})$, one can be interested in controlling the norm in $L^\infty(\Xi; \mathcal{V})$

$$\|u - u_\Lambda\|_{L^\infty(\Xi;\mathcal{V})} = \sup_{\xi \in \Xi} \|u(\xi) - u_\Lambda(\xi)\|_{\mathcal{V}}$$

  Let $u = \sum_\alpha v_\alpha \varphi_\alpha$ with $\varphi_\alpha = \psi_\alpha / \|\psi_\alpha\|_{L^\infty}$, then

$$\|u - u_\Lambda\|_{L^\infty(\Xi;\mathcal{V})} \leq \sum_{\alpha \notin \Lambda} \|v_\alpha\|_{\mathcal{V}}$$

- Best n-term approximation can be defined in the L^∞ norm

$$\sigma_n^{(\infty)} = \min_{\#\Lambda_n = n} \min_{v \in X_{\Lambda_n}} \|u - v\|_{L^\infty(\Xi;\mathcal{V})}$$

- If $(\|v_\alpha\|_{\mathcal{V}})_\alpha \in \ell^p$ with $p < 1$, then there exists Λ_n with $\#\Lambda_n = n$ such that

$$\|u - u_{\Lambda_n}\|_{L^\infty(\Xi;\mathcal{V})} \leq C n^{-s}, \quad C = \|(\|v_\alpha\|_{\mathcal{V}})_\alpha\|_{\ell^p}, \quad s = \frac{1}{p} - 1$$

  so that

$$\sigma_n^{(\infty)} \leq C n^{-s}$$

- See [Cohen-DeVore-Schwab 2010,Chkifa-Cohen-Schwab 2014] for a proof of $(\|v_\alpha\|_{\mathcal{V}})_\alpha \in \ell^p$ for a large class of problems.

Quasi-optimal index sets

- Assume that we know a bound $\delta(\alpha)$ such that

$$\|u_\alpha\|_{\mathcal{V}} \leq \delta(\alpha) \quad (1)$$

- Quasi-optimal index set Λ_n^δ obtained by retaining the n largest values $\delta(\alpha)$. Close to the optimal if the bound (1) is sharp.
- Estimation of $\delta(\alpha)$ based on a priori analysis (a priori definition of the sequence Λ_n^δ) or based on a posteriori analysis (adaptive construction).
- Suppose that there exists $\gamma \geq 1$ such that

$$\gamma^{-1}\delta(\alpha) \leq \|u_\alpha\|_{\mathcal{V}} \leq \delta(\alpha)$$

Then

$$\|u - u_{\Lambda_n^\delta}\|^2_{L^2_\mu(\Xi;\mathcal{V})} = \sum_{\alpha \notin \Lambda_n^\delta} \|u_\alpha\|^2_{\mathcal{V}} \leq \sum_{\alpha \notin \Lambda_n^\delta} \delta(\alpha)^2 = \min_{\#\Lambda_n = n} \sum_{\alpha \notin \Lambda_n} \delta(\alpha)^2 \leq \gamma^2 \min_{\#\Lambda_n = n} \sum_{\alpha \notin \Lambda_n} \|u_\alpha\|^2_{\mathcal{V}}$$

and therefore

$$\|u - u_{\Lambda_n^\delta}\|_{L^2_\mu(\Xi;\mathcal{V})} \leq \gamma \sigma_n^{(2)} \quad \text{(Quasi-optimality)}$$

Quasi-optimal index sets based on a priori analysis

- In practice, define a sequence of subsets

$$\Lambda_p = \{\alpha : \delta(\alpha) \geq \epsilon(p)\}$$

with $(\epsilon(p))_{p\geq 0}$ a decreasing sequence.

- Assume

$$\|u_\alpha\|_{\mathcal{V}} \leq Ce^{-\sum_k \omega_k \alpha_k} := \delta(\alpha) \qquad (\star)$$

If C independent of α, take $\epsilon(p) = Ce^{-p}$, so that

$$\Lambda_p = \left\{\alpha : \sum_k \omega_k \alpha_k \leq p\right\} \quad \text{(Anisotropic total degree)}$$

If $C = C(\alpha)$, take $\epsilon(p) = e^{-p}$, so that

$$\Lambda_p = \left\{\alpha : \sum_k \omega_k \alpha_k - \log(C(\alpha)) \leq p\right\} \quad \text{(modified Anisotropic total degree)}$$

- See [Back-Nobile-Tamellini-Tempone 2011, 2012, 2014] for a proof of $(\star)$ for some classes of parametric problems.

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Tensor spaces

An algebraic tensor space $V = V_1 \otimes \ldots \otimes V_d$ is the set of elements of the form

$$u = \sum_{i=1}^{m} v_i^1 \otimes \ldots \otimes v_i^d$$

or for multivariate functions

$$u(x_1, \ldots, x_d) = \sum_{i=1}^{m} v_i^1(x_1) \ldots v_i^d(x_d).$$

A tensor Banach space $V_{\|\cdot\|}$ is obtained by the completion of the algebraic tensor space V with respect to a norm $\|\cdot\|$:

$$V_{\|\cdot\|} = \overline{V_1 \otimes \ldots \otimes V_d}^{\|\cdot\|}.$$

W. Hackbusch.
Tensor Spaces and Numerical Tensor Calculus,
Springer, 2012.

Examples of (Banach) tensor spaces

- Multidimensional array

$$a \in \mathbb{R}^{n_1 \times \ldots \times n_d} = \mathbb{R}^{n_1} \otimes \ldots \otimes \mathbb{R}^{n_d}$$

$$a = \sum_{i_1=1}^{n_1} \ldots \sum_{i_d=1}^{n_d} a_{i_1,\ldots,i_d} e^1_{i_1} \otimes \ldots \otimes e^d_{i_d}$$

- Finite dimensional tensor spaces:

$$V = V_1 \otimes \ldots \otimes V_d = V_{\|\cdot\|}$$

Denoting $\{\phi_i^k\}_{i=1}^{n_k}$ a basis of the n_k-dimensional space V_k, $u \in V$ can be written

$$u = \sum_{i_1=1}^{n_1} \ldots \sum_{i_d=1}^{n_d} a_{i_1 \ldots i_d} \phi^1_{i_1} \otimes \ldots \otimes \phi^d_{i_d},$$

and identified with

$$a \in \mathbb{R}^{n_1 \times \ldots \times n_d} = \mathbb{R}^{n_1} \otimes \ldots \otimes \mathbb{R}^{n_d}$$

Examples of (Banach) tensor spaces

- Lebesgue space $L^p_\mu(\Xi)$ with product measure $\mu = \mu_1 \otimes \ldots \otimes \mu_d$ on $\Xi = \Xi_1 \times \ldots \times \Xi_d$:

$$L^p_\mu(\Xi_1 \times \ldots \times \Xi_d) = \overline{L^p_{\mu_1}(\Xi_1) \otimes \ldots \otimes L^p_{\mu_1}(\Xi_d)}^{\|\cdot\|_p} \quad (1 \leq p < \infty)$$

An element $u \in L^p_{\mu_1}(\Xi_1) \otimes \ldots \otimes L^p_{\mu_1}(\Xi_d)$ is of the form

$$u(\xi_1, \ldots, \xi_d) = \sum_{i=1}^m u_i^1(\xi_1) \ldots u_i^d(\xi_d), \quad (\xi_1, \ldots, \xi_d) \in \Xi.$$

- Sobolev space $W^{s,p}(I)$ on $I = I_1 \times \ldots \times I_d$, the set of measurable functions $u : I \to \mathbb{R}$ with bounded norm

$$\|u\|_{s,p} = \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_p, \quad \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}.$$

$$W^{s,p}(I) = \overline{W^{s,p}(I_1) \otimes \ldots \otimes W^{s,p}(I_d)}^{\|\cdot\|_{s,p}} \quad (1 \leq p < \infty)$$

$W^{s,p}(I)$ is an intersection tensor space:

$$W^{s,p}(I) = \bigcap_{\alpha \in \Lambda_s} \overline{W^{\alpha_1,p} \otimes \ldots \otimes W^{\alpha_d,p}}^{\|\cdot\|_\alpha}$$

$$\Lambda_s = \{(0, \ldots, 0), (s, 0, \ldots, 0), (0, \ldots, 0, s)\}$$

Tensors in stochastic and parametric problems

- Stochastic/Parametric equations (PDEs, ODEs...):

$$\mathcal{F}(u(\xi);\xi) = 0, \quad u(\xi) \in \mathcal{V}$$

$$\xi \sim \mu, \quad supp(\mu) = \Xi.$$

$$u \in L^p_\mu(\Xi;\mathcal{V}) = \overline{\mathcal{V} \otimes L^p_\mu(\Xi)}$$

- Functions of independent random variables:

$$u(\xi_1, \xi_2, \dots, \xi_d)$$

$$\xi_k \sim \mu_k, \quad supp(\mu_k) = \Xi_k$$

$$u \in \overline{\mathcal{V} \otimes L^p_\mu(\Xi)} = \overline{\mathcal{V} \otimes L^p_{\mu_1}(\Xi_1) \otimes \dots \otimes L^p_{\mu_d}(\Xi_d)}$$

- Parametrized functions of random variables (robust optimization and control, statistical inverse problems):

$$u(\xi, \eta)$$

$$\xi \sim \mu, \quad supp(\mu) = \Xi, \quad \eta \in A$$

$$u \in \overline{\mathcal{V} \otimes L^p_\mu(\Xi) \otimes L^q_\nu(A)}$$

- Stochastic differential equations:

$$\begin{cases} dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t \\ X_0 = x_0 \end{cases} \qquad X_t = (X_t^1 \dots X_t^n)$$

  - The probability density function $u(\cdot, t)$ of X_t verifies a n-dimensional PDE (Kolmogorov forward equation)

$$u(\cdot, t) \in \overline{H^1_{\mu_1}(\mathbb{R}) \otimes \dots \otimes H^1_{\mu_n}(\mathbb{R})}$$

  - Approximation of Wiener process $W_t \approx \sum_{k=1}^d \varphi_k(t)\xi_k$, and

$$X_t \approx X_t(\xi_1, \dots, \xi_d) \in \overline{L^2_\Gamma(\mathbb{R}) \otimes \dots \otimes L^2_\Gamma(\mathbb{R})}$$

Notions of ranks for higher-order tensors

- For order-two tensors, a single notion of rank.

$$rank(v) \leq r \iff v = \sum_{i=1}^{r} v_i^1 \otimes v_i^2 \quad \left(v(x_1, x_2) = \sum_{i=1}^{r} v_i^1(x_1) v_i^2(x_2) \right)$$

- For higher-order tensors, different notions of rank.

- Canonical rank:

$$rank(v) \leq r \iff v = \sum_{i=1}^{r} v_i^1 \otimes \ldots \otimes v_i^d \quad \text{or} \quad v(x) = \sum_{i=1}^{r} v_i^1(x_1) \ldots v_i^d(x_d)$$

Parametrization with $\sum_{\nu=1}^{d} \dim(V_\nu) = O(d)$ parameters.

Example

Ishigami function

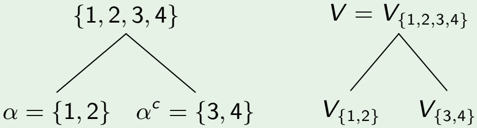
$$v(x_1, x_2, x_3) = \sin(x_1) + a\sin(x_2)^2 + bx_3^4 \sin(x_1) = \sin(x_1)(1 + bx_3^4) + a\sin(x_2)^2$$

has a canonical rank 2.

- α-rank:
  for $\alpha \subset \{1,\dots,d\}$, $V = V_\alpha \otimes V_{\alpha^c}$, with $V_\alpha = \bigotimes_{\mu\in\alpha} V_\mu$ and define the α-rank:

$$rank_\alpha(v) \le r_\alpha \quad \Longleftrightarrow \quad v = \sum_{i=1}^{r_\alpha} v_i^\alpha \otimes v_i^{\alpha^c}, \quad v_i^\alpha \in V_\alpha, \quad v_i^{\alpha^c} \in V_{\alpha^c}$$

Example



$u(x_1,\dots,x_4) = f(x_1,x_2)g(x_3,x_4)$ is such that $rank_{(1,2)}(u) = rank_{(3,4)}(u) = 1$.

- Relation with minimal subspaces:
  The minimal subspace $U_\alpha^{min}(v)$ of $v \in V$ is the smallest subspace in V_α such that

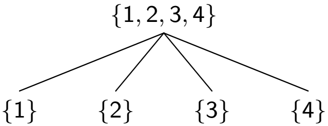
$$v \in U_\alpha^{min}(v) \otimes V_{\alpha^c}$$

$$\boxed{rank_\alpha(v) = dim(U_\alpha^{min}(v))}$$

For $v = \sum_{i=1}^{r_\alpha} v_i^\alpha \otimes v_i^{\alpha^c}$, $U_\alpha^{min}(v) = span\{v_i^\alpha : 1 \le i \le r_\alpha\}$.

- Tucker rank:

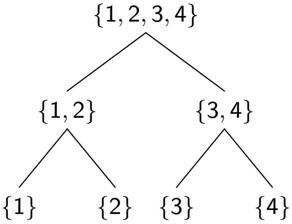
$$rank_T(v) = (rank_1(v), \dots, rank_d(v))$$



$$rank_T(v) \leq r = (r_1, \dots, r_d) \iff v = \sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_d} a_{i_1 \dots i_d} v^1_{i_1} \otimes \dots \otimes v^d_{i_d}$$

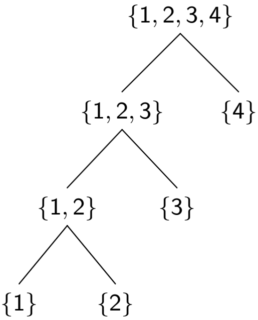
- Tree-based Tucker rank:

$$rank_T(v) = (rank_\alpha(v); \alpha \in T) \quad \text{with } T \text{ a dimension tree}$$



- TT-rank:

$$rank_{TT}(v) = (rank_{\{1\}}(v), rank_{\{1,2\}}(v) \dots, rank_{\{1,\dots,d-1\}}(v))$$



$$rank_{TT}(v) \leq r = (r_1, \dots, r_{d-1}) \iff v = \sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_{d-1}} v^1_{1,i_1} \otimes v^2_{i_1,i_2} \otimes \dots \otimes v^d_{i_{d-1},1}$$

Example

$v(x) = v_1(x_1) + \dots + v_d(x_d)$ has a tree based rank $(2, \dots, 2)$.

Low-rank tensor subsets

- Different notions of rank yield different low-rank tensor subsets: Canonical, Tucker, Tree-based Tucker (HT, TT), ...

$$\mathcal{M}_{\leq r} = \{v \in V : rank(v) \leq r\}$$

- $\mathcal{M}_{\leq r}$ has a small dimension $n(r, d)$ (i.e. can be parameterized with a small number n of parameters), typically

$$n(r, d) = O(dr^s)$$

Geometry of low-rank tensors subsets

See [Holtz-Rohwedder-Schneider 11, Uschmajew-Vandereycken 13] for Hilbert setting. See [Falco-Hackbusch-Nouy 14] for a Banach setting and for general tree-based format.

- Subsets of tensors with fixed tree-based rank have a manifold structure :

$$\mathcal{M}_{\leq r} = \bigcup_{s \leq r} \mathcal{M}_{=s}$$

$$\mathcal{M}_{=s} = \{v \in V : rank_T(v) = s\} = \left\{ v = F_{\mathcal{M}}(p) \; ; \; p = (p_1, \ldots, p_L) \in \mathcal{P}^1 \times \ldots \times \mathcal{P}^L \right\}$$

  where $F_{\mathcal{M}}$ is a multilinear map and the $\mathcal{P}^l$ are low-dimensional vector spaces (or manifolds).

  - $\mathcal{M}_{=r}$ is an analytic Banach manifold, with explicit local charts.
  - Under the same assumptions as before on the norms $\{\| \cdot \|_\alpha \colon \alpha \in T\}$, $\mathcal{M}_{=r}$ is an embedded submanifold.

- Interesting consequences:
  - Optimization algorithms on manifolds
  - Dynamical systems on low-rank manifolds (Dirac-Frenkel can be extended to topological tensor spaces)

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Approximation in low-rank subsets

- Approximation of a high order tensor

$$u \in \overline{V_1 \otimes \ldots \otimes V_d}^{\|\cdot\|}$$

in a subset of tensors with bounded rank

$$\mathcal{M}_{\leq r} = \{v \in V_1 \otimes \ldots \otimes V_d : rank(v) \leq r\}$$

- For all tensor formats, since $\cup_r \mathcal{M}_{\leq r}$ is dense in the tensor space, then

$$\inf_{\mathcal{M}_{\leq r}} \|u - v\|$$

converges to zero when $r \to \infty$.
- Questions :
  - How fast does it converge to zero ?
  - Do we beat the curse of dimensionality ?
  - Existence of best approximations ?

Approximation in low-rank tensor subsets (canonical format)

- Good approximation for smooth functions [Temlyakov, Uschmajew-Schneider]

Example (Sobolev regularity: approximation in canonical format)

$$\inf_{v \in \mathcal{R}_r} \|u - v\|_{L^2} \lesssim r^{-sd/(d-1)} \quad \forall u \in B^s_{mix} \subset L^2(\pi_d)$$

$$\text{with} \quad B^s_{mix} = \left\{ u \in L^2(\pi_d); \|u\|_{H^s_{mix}} \leq 1 \right\} \subset H^s_{mix}(\pi_d)$$

That means that for any $u \in B^s_{mix}$ and for $\epsilon > 0$, it could be possible to find an approximation $v(x_1, \ldots, x_d) = \sum_{i=1}^r \phi_i^1(x_1) \ldots \phi_i^d(x_d)$ such that

$$\|u - v\| \leq \epsilon \quad \text{with} \quad r \gtrsim \epsilon^{-\frac{d-1}{sd}}$$

- But low-rank approximations are expected to exploit additional features.

Sparse tensor approximation

Suppose $\{\phi_i^\nu(x_\nu) : i \in \Lambda_\nu\}$ is a basis of V_ν (e.g. polynomial spaces). Then

$$\{\phi_{i_1}^1(x_1)\dots\phi_{i_d}^d(x_d) : (i_1,\dots,i_d) \in \Lambda = \Lambda_1 \times \dots \times \Lambda_d\}$$

is a basis of $V_1 \otimes \dots \otimes V_d$. An element $v \in V_1 \otimes \dots \otimes V_d$ can be written

$$v(x_1,\dots,x_d) = \sum_{i\in\Lambda} a_i \phi_{i_1}^1(x_1)\dots\phi_{i_d}^d(x_d)$$

A sparse tensor approximation v_N of v is under the form

$$v_N(x_1,\dots,x_d) = \sum_{i\in\Lambda_N} a_i \phi_{i_1}^1(x_1)\dots\phi_{i_d}^d(x_d)$$

$$\Lambda_N \subset \Lambda, \quad \#\Lambda_N = N \ll \#\Lambda$$

and has a canonical rank

$$rank(v_N) \leq N$$

Consequence: if the best N-term approximation v_N (for a fixed basis) is such that $\|v - v_N\| = \sigma_N$, then

$$\inf_{v_N \in \mathcal{R}_N} \|v - v_N\| \leq \sigma_N$$

Approximation of rank one functions

Consider the function $u : [0,1]^d \to \mathbb{R}$

$$u(x_1, \dots, x_d) = u_1(x_1) \dots u_d(x_d), \quad u_k \in C^s(0,1)$$

- Approximation of the factors

$$u_k(x_k) \approx \sum_{i=1}^{n} c_{k,i} \varphi_i(x_k) \quad \text{with error} \sim n^{-s}$$

- Number of parameters

$$N = dn$$

- Number of parameters to achieve accuracy ϵ

$$N(\epsilon, d) = O(d^{1+1/s} \epsilon^{-1/s})$$

Complexity of approximating rank-one tensors

- Consider the set of functions [Bachmayr 2013,Novak 2014]

$$F_M = \left\{ u(x_1, \dots, x_d) = \prod_{i=1}^{d} u_i(x_i) : \|u_i\|_\infty \leq 1, \ \|u_i^{(s)}\|_\infty \leq M \right\}$$

  - If $M \geq 2^s s!$,

$$N(\epsilon, d) \geq 2^d \quad \text{for all } \epsilon < 1 \quad (\text{curse of dimensionality})$$

  - Polynomial tractability (algorithm available) for the sets of functions

$$F_M^{x^*} = \{u \in F_M : u(x^*) \neq 0 \quad \text{for a known } x^*\}$$

$$F_M^V = \{u \in F_M : u \neq 0 \text{ on a box of measure greater than } V\}$$

Approximation with low-rank tensors (canonical format)

- Consider that function $u : [0,1]^d \to \mathbb{R}$ can be approximated with accuracy ϵ by

$$u(x_1, \dots, x_d) \approx \sum_{i=1}^{r(\epsilon)} u_{1,i}(x_1) \dots u_{d,i}(x_d), \quad u_{k,i} \in C^s(0,1)$$

  - Approximation of each factor $u_{k,i}$ using n parameters with accuracy $\sim n^{-s}$
  - Number of parameters $N = r(\epsilon) nd$
  - Number of parameters to achieve accuracy ϵ

$$N(\epsilon, d) = O(r(\epsilon)^{1/s} d^{1+1/s} \epsilon^{-1/s})$$

- Do we beat the curse of dimensionality ?
  - What about $r(\epsilon)$ with respect to d ?
  - Which information on u and which algorithm ?

Best approximation in low-rank tensor subsets : canonical format

- If $\|\cdot\|$ is stronger than the injective norm, then the set of rank-one tensors $\mathcal{R}_1$ is weakly closed and best rank-one approximation problem is well-posed

$$\inf_{v \in \mathcal{R}_1} \|u - v\|$$

- For $d \geq 3$ and $r \geq 2$, the set $\mathcal{R}_r$ of tensors with canonical rank bounded by r is not closed, and then the best approximation problem in canonical format

$$\inf_{v \in \mathcal{R}_r} \|u - v\|$$

is ill posed.

Best approximation in low-rank tensor subsets: tree-based formats

- Best approximation problems in tree-based low-rank subsets $\mathcal{M}_{\leq r}$ are well-posed for any r, provided some conditions on tensor norms.

Theorem [Falco-Hackbusch-Nouy 14]

Assume that for all nodes $\alpha \in T$ with sons $S(\alpha) \neq \emptyset$,

- $V_\alpha = \bigotimes_{\beta \in S(\alpha)} V_\beta$ equipped with a norm $\|\cdot\|_\alpha$
- $\bigotimes : \times_{\beta \in S(\alpha)} (\overline{V_\beta}^{\|\cdot\|_\beta}, \|\cdot\|_\beta) \to \left(\bigotimes_{\beta \in S(\alpha)} \overline{V_\beta}^{\|\cdot\|_\beta}, \|\cdot\|_\alpha \right)$ is continuous
- $\|\cdot\|_\alpha$ is stronger than the injective norm $\|\cdot\|_{\vee(S(\alpha))}$ induced by the norms $\{\|\cdot\|_\beta : \beta \in S(\alpha)\}$.

Then $rank_T(\cdot) : \overline{V}^{\|\cdot\|_D} \to \mathbb{R}^{\#T}$ is weakly l.s.c.. Therefore $\mathcal{M}_{\leq r}$ is weakly closed.

- Theorem directly applies for L^p spaces.
- For Sobolev spaces $W^{s,p}$, the theorem can be used indirectly, by writing $W^{s,p}$ as an intersection of tensor spaces for which the conditions hold.

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Alternating minimization algorithm

- Parametrization of low-rank subsets

$$\mathcal{M}_{\leq r} = \left\{ v = F(p) \ ; \ p = (p_1, \ldots, p_L) \in \mathcal{P}^1 \times \ldots \times \mathcal{P}^L \right\}$$

  with F a multilinear map.

- Best approximation problem

$$\inf_{\mathcal{M}_{\leq r}} J(v) = \inf_{p_1, \ldots, p_L} J(F(p_1, \ldots, p_L))$$

- Alternating minimization algorithm: solve successively

$$\min_{p_k} J(F(p_1, \ldots, p_k, \ldots, p_L)) \qquad (\star)$$

  - If J is convex, then the partial map $G_k : p_k \mapsto J(F(p_1, \ldots, p_k, \ldots, p_L))$ is also convex.
  - If J is quadratic, then G_k is quadratic and $(\star)$ is a quadratic optimization problem.
  - Under standard assumption on J, convergence to a stationary point can be proved but no guaranty for obtaining the global optimum (except for $d = 2$ and $\mathcal{J}(v) = \|u - v\|$ with $\| \cdot \|$ a canonical inner product norm).

Quasi-best approximations (canonical inner product norm and tree-based formats)

- Consider Hilbert tensor spaces with induced canonical norms.
- For tree-based tensor formats, algorithms based on higher-order versions of SVD for obtaining quasi-best approximations

$$u_r \in \mathcal{M}_{\leq r} \quad \text{such that} \quad \|u - u_r\| \leq \gamma(d) \inf_{v \in \mathcal{M}_{\leq r}} \|u - v\|$$

$$\gamma(d) = \begin{cases} \sqrt{d} & \text{for Tucker tensors} \\ \sqrt{2d-2} & \text{for tree-based Tucker tensors} \end{cases}$$

- Higher-order SVD for Tucker format [De Lathauwer et al 2000]
  - Truncated SVD of matricisation of $u \in V_k \otimes V_{[k]}$:

$$u \approx u^k_{r_k} = \sum_{i=1}^{r_k} v^k_i \otimes v^{[k]}_i$$

  - Subspaces $U^k_{r_k} = span\{v^k_1 \dots v^k_{r_k}\}$.
  - Approximation u_r defined by $U^1_{r_1} \otimes \dots \otimes U^d_{r_d}$

$$\|u - u_r\| = \min_{v \in U^1_{r_1} \otimes \dots \otimes U^d_{r_d}} \|u - v\|$$

  and such that

$$u_r = (P^1_{r_1} \otimes \dots \otimes P^d_{r_d})u$$

  where $P^k_{r_k}$ is the orthogonal projector from V^k onto $U^k_{r_k}$.

- Higher-order SVD for tree-based format [Grasedyck 2010]
  - Truncated SVD of matricisations of $u \in V_\alpha \otimes V_{\alpha^c}$
  - Subspaces $U^\alpha_{r_\alpha} \subset V_\alpha$.
  - Hierarchical composition of projections.

Approximation of higher-order tensors in canonical format

- Optimization in canonical format is ill-posed

$$\inf_{v \in \mathcal{R}_r} \mathcal{E}(u, v)$$

- A quasi-optimal approximation $u_r \in \mathcal{R}_r$ could (in principle) be obtained

$$\mathcal{E}(u, u_r) \leq (1 + \epsilon) \inf_{v \in \mathcal{R}_r} \mathcal{E}(u, v)$$

  but usually numerically unstable
- No notion of decomposition

$$u_r = \sum_{i=1}^{r} u_i^{(1,r)} \otimes \ldots \otimes u_i^{(d,r)}$$

Greedy approximation in canonical format (Proper Generalized Decomposition)

- Suboptimal construction of canonical representation using greedy algorithms.
- Starting from $u_0 = 0$, then
$$u_m = u_{m-1} + w_m$$
with $w_m = w_m^1 \otimes \ldots \otimes w_m^d$ defined by
$$\mathcal{E}(u, u_{m-1} + w_m) = \min_{w \in \mathcal{R}_1} \mathcal{E}(u, u_{m-1} + w)$$
- Notion of decomposition
$$u_m = \sum_{i=1}^{m} w_i^1 \otimes \ldots \otimes w_i^d$$
- Possible optimization of functions after each correction for improving convergence (but we loose the notion of decomposition).

Greedy approximation (Proper Generalized Decomposition)

- Possible corrections w_m in other low-rank subsets.
- Convergence results available [Cances & al 2011, Falco & N. 2012] (not really specific to tensor setting) but no a priori estimates (except for very general and pessimistic results).
- Convergence may be slow compared to $\inf_{w \in \mathcal{R}_r} \mathcal{E}(u, w)$
- The construction does not really exploit the tensor structure.

Greedy approximation in Tucker format: a subspace point of view

- Tucker tensors with bounded multilinear rank:

$$\mathcal{T}_r = \{v : rank_\mu(v) \leq r_\mu, \forall \mu\}$$
$$= \left\{ v \in \bigotimes_{\mu=1}^{d} U_\mu \;:\; dim(U_\mu) \leq r_\mu, \forall \mu \right\}$$

- Best approximation — a subspace point of view:

$$\min_{v \in \mathcal{T}_r} \mathcal{E}(u, v) = \min_{dim(U_1)=r_1} \dots \min_{dim(U_d)=r_d} \min_{v \in U_1 \otimes \dots \otimes U_d} \mathcal{E}(u, v)$$

  This yields sequences of optimal but non necessarily nested subspaces $\{U_\mu^{r_\mu} : r_\mu \geq 1\}$.

- Greedy construction of subspaces with nestedness property

$$\mathcal{E}(u, u_m) = \min_{U_1^m \supset U_1^{m-1}} \dots \min_{U_d^m \supset U_d^{m-1}} \min_{v \in U_1^m \otimes \dots \otimes U_d^m} \mathcal{E}(u, v)$$

  Constructive approach. Possible introduction of anisotropic enrichment.

- Possible (tricky!) extension to tree-based tensor formats.

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Sample-based low-rank tensor approximation

- Computation of an output variable of interest

$$s(\xi) = \ell(u(\xi); \xi), \quad \xi \sim \mu,$$

$$s \in L^2_\mu(\mathbb{R}^d)$$

- Best L^2 approximation in a low-rank subset $\mathcal{M}$ (not computable)

$$\min_{v \in \mathcal{M}} \|s - v\|^2 = \min_{v \in \mathcal{M}} \mathbb{E}_\mu((s(\xi) - v(\xi))^2)$$

- Suppose we have evaluations $\{s(\xi^k)\}_{k=1}^K$ of s at sample points $\{\xi^k\}_{k=1}^K$ (Simulations of the full order model or of a reduced order model).
- Low rank approximation using least-square minimization:

$$\min_{v \in \mathcal{M}} \|s - v\|_K^2$$

$$\text{with} \quad \|s - v\|_K^2 = \frac{1}{K} \sum_{k=1}^{K} (s(\xi^k) - v(\xi^k))^2 \approx \mathbb{E}_\mu((s(\xi) - v(\xi))^2)$$

Sample-based low-rank tensor approximation

- Regularization could be required

$$\min_{v\in\mathcal{M}} \|s - v\|_K^2 + \text{“regularization”}$$

- For a given tensor format

$$\mathcal{M} = \{v = F_{\mathcal{M}}(p_1, \dots, p_L); p_k \in \mathbb{R}^{m_k}, 1 \le k \le L\}$$

solve

$$\min_{p_1,\dots p_L} \|s - F_{\mathcal{M}}(p_1, \dots, p_L)\|_K^2 + \sum_k \lambda_k \|p_k\|_s$$

that corresponds to a minimization in a subset of $\mathcal{M}$:

$$\mathcal{M}_\gamma = \{v = F_{\mathcal{M}}(p_1, \dots, p_L); p_k \in \mathbb{R}^{m_k}, \|p_k\|_s \le \gamma_k, 1 \le k \le L\}$$

-
- Sparsity-inducing regularization with $0 \le s \le 1$.
- Some references [Beylkin-Garcke-Mohlenkamp 2011; Doostan-Validi-Iaccarino 2013; Chevreuil-Lebrun-Nouy-Rai 2014]

Now, entering the model...

Higher order tensor structure

- Suppose that $\mu = \mu_1 \otimes \ldots \otimes \mu_d$, a product measure on $\Xi_1 \times \ldots \times \Xi_d$ (e.g. when $\xi = (\xi_1, \ldots, \xi_d)$ are independent random variables). Then
$$L^2_\mu(\Xi) = \overline{L^2_{\mu_1}(\Xi_1) \otimes \ldots \otimes L^2_{\mu_d}(\Xi_d)}$$
- Suppose that the approximation space $\mathcal{S} \subset L^2_\mu(\Xi)$ is a finite dimensional tensor space
$$\mathcal{S} = \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_d, \quad \mathcal{S}_\nu \subset L^2_{\mu_\nu}(\Xi_\nu)$$
and the same for $\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}_1 \otimes \ldots \otimes \widetilde{\mathcal{S}}_d$.
- $\lambda^{(\nu)} : \Xi_\nu \to \mathbb{R}$ can be identified with an operator $\Lambda^{(\nu)} : \mathcal{S}_\nu \to \widetilde{\mathcal{S}}'_\nu$ such that
$$\langle \Lambda^{(\nu)} s, \tilde{s} \rangle = \mathbb{E}_{\mu_\nu}(\lambda^{(\nu)}(\xi_\nu) s(\xi_\nu) \tilde{s}(\xi_\nu))$$
- A function $\lambda : \Xi \to \mathbb{R}$ such that $\lambda(\xi) = \lambda^{(1)}(\xi_1) \ldots \lambda^{(d)}(\xi_d)$ can be identified with an operator $\Lambda : \mathcal{S} \to \widetilde{\mathcal{S}}'$ such that
$$\Lambda = \Lambda^{(1)} \otimes \ldots \otimes \Lambda^{(d)}$$

- Equation

$$A(\xi)u(\xi) = f(\xi)$$

- Suppose that

$$A(\xi) = \sum_{k=1}^{R} A_k \lambda_k(\xi), \quad \text{with} \quad \lambda_k(\xi) = \lambda_k^{(1)}(\xi_1)\dots\lambda_k^{(d)}(\xi_d)$$

and

$$f(\xi) = \sum_{k=1}^{L} f_k \eta_k(\xi), \quad \text{with} \quad \eta_k(\xi) = \eta_k^{(1)}(\xi_1)\dots\eta_k^{(d)}(\xi_d)$$

- Tensor structured equation for $u \in \mathcal{V} \otimes \mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_d$

$$Bu = F \quad \Longleftrightarrow \quad \left(\sum_{k=1}^{R} B_k \otimes \Lambda_k^{(1)} \otimes \dots \otimes \Lambda_k^{(d)}\right) u = \sum_{k=1}^{L} f_k \otimes \eta_k^{(1)} \otimes \dots \otimes \eta_k^{(d)}$$

- Tensor structured equation in algebraic form for $\mathbf{u} \in \mathbb{R}^{\mathcal{N}} \otimes \mathbb{R}^{\mathcal{P}_1} \otimes \dots \otimes \mathbb{R}^{\mathcal{P}_d}$

$$\mathbf{Bu} = \mathbf{F} \quad \Longleftrightarrow \quad \left(\sum_{k=1}^{R} \mathbf{A}_k \otimes \boldsymbol{\Lambda}_k^{(1)} \otimes \dots \otimes \boldsymbol{\Lambda}_k^{(d)}\right) \mathbf{u} = \sum_{k=1}^{L} \mathbf{f}_k \otimes \boldsymbol{\eta}_k^{(1)} \otimes \dots \otimes \boldsymbol{\eta}_k^{(d)}$$

Classical iterative methods with low-rank truncations

- Equation in tensor format

$$Bu = F, \quad u \in \mathcal{V} \otimes \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_d$$

- Iterative solver

$$u^{(k)} = T(u^{(k-1)}) \quad (T\text{: iteration map})$$

- Approximate iterations using low-rank truncations:

$$u^{(k)} \in \mathcal{M}_{r(\epsilon)} \quad \text{such that} \quad \|u^{(k)} - T(u^{(k-1)})\| \leq \epsilon$$

- For the canonical norm $\|\cdot\|$, truncation based on higher-order SVD
- Computational requirements: low-rank algebra and efficient truncation algorithms
- Analysis : perturbation of algorithms.

(see e.g. [Khoromskij-Schwab 2011, Kressner-Tobler 2011])

Minimal residual low-rank approximation

- Residual-based error

$$\mathcal{E}(u, w) = \|Bw - F\|_C = \|w - u\|_{B^*CB}$$

with a certain residual norm $\|\cdot\|_C^2 = \langle C\cdot, \cdot\rangle$.

- Best approximation in $\mathcal{M}_{\leq r}$

$$\mathcal{E}(u, u_r) = \min_{w \in \mathcal{M}_{\leq r}} \mathcal{E}(u, w)$$

- If

$$\alpha\|u - w\| \leq \mathcal{E}(u, w) \leq \beta\|u - w\|$$

then

$$\|u - u_r\| \leq \frac{\beta}{\alpha} \min_{w \in \mathcal{M}_{\leq r}} \|u - w\|$$

Illustration : stationary advection-diffusion-reaction equation

$$-\nabla \cdot (\kappa \nabla u) + c \cdot \nabla u + \gamma u = \delta I_{\Omega_1}(x) \quad on \quad \Omega$$

Random field

$$\kappa(x, \boldsymbol{\xi}) = \mu_\kappa + \sum_{i=1}^{40} \sqrt{\sigma_i} \kappa_i(x) \xi_i, \quad \xi_i \in U(-1, 1)$$

Spatial modes $\kappa_i(x)$

Amplitudes σ_i

Stochastic approximation

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{40}), \quad \boldsymbol{\Xi} = (-1,1)^{40} = \Xi_1 \times \ldots \times \Xi_{40}$$

$$\mathcal{S} = \mathbb{P}_4(\Xi_1) \otimes \ldots \otimes \mathbb{P}_4(\Xi_{40})$$

$$\boxed{dim(\mathcal{S}) = 5^{40} \approx 10^{28}}$$

Finite element mesh

$$\boxed{dim(\mathcal{V}_N) = 4435}$$

Solution $u(\cdot, \boldsymbol{\mu}_{\boldsymbol{\xi}})$ for mean parameters

A basic hierarchical format

Deterministic/stochastic separation

$$u(\xi) \approx u_m(\xi) = \sum_{i=1}^{m} v_i s_i(\xi)$$

$$\hookrightarrow \quad \boxed{\mathcal{V}_m = span\{v_i\}_{i=1}^m}$$

low-rank approximation of parametric functions

$$\mathbf{s}(\xi) := (\mathbf{s}_i)_{i=1}^m \approx \mathbf{s}_Z(\xi) = \sum_{k=1}^{Z} \phi_k^0 \prod_{j=1}^{d} \phi_k^j(\xi_j)$$

$$\hookrightarrow \quad \boxed{\mathcal{S}_Z = span\{\prod_{j=1}^d \phi_k^j(\xi_j)\}_{k=1}^Z}$$

For a precision $\|u - u_{M,Z}\|_{L^2} \leqslant 10^{-2}$

- $\boxed{dim(\mathcal{V}_m) \approx 15} \ll 4435 = dim(\mathcal{V})$
- $\boxed{dim(\mathcal{S}_Z) \approx 10} \ll 10^{28} = dim(\mathcal{S})$
- 15 classical deterministic problems in order to build $\mathcal{V}_M \subset \mathcal{V}_N$
- about 1 minute computation on a laptop with matlab

Convergence properties of quantities of interest

Probability of events

Quantity of interest

$$Q(\boldsymbol{\xi}) = \int_{\Omega_2} u(x, \boldsymbol{\xi})\, dx$$

$$Q_M(\boldsymbol{\xi}) = \int_{\Omega_2} u_M(x, \boldsymbol{\xi})\, dx$$

$$P(Q > q), \quad q \in (3.5, 5.4)$$

Convergence properties of quantities of interest

Sensitivity analysis

$$Q(\boldsymbol{\xi}) \approx Q_M(\boldsymbol{\xi}) \approx Q_{M,Z}(\boldsymbol{\xi}) = \sum_{k=1}^{Z} q_k \Psi_k(\boldsymbol{\xi}), \quad \Psi_k(\boldsymbol{\xi}) = \prod_{i=1}^{40} \phi_k^i(\xi_i)$$

First order Sobol sensitivity index with respect to parameter ξ_i

$$\boxed{S_i = \frac{Var(E(Q|\xi_i))}{Var(Q)}} \quad E(Q|\xi_i) = \sum_{k=1}^{Z} \alpha_k^i \phi_k^i(\xi_i), \quad \alpha_k^i = q_k \prod_{\substack{j=1 \\ j\neq i}}^{40} E(\phi_k^i(\xi_i))$$

First order Sobol sensivity indices S_i

M=1 M=5 M=15

Challenging issues

- Classify applications and dedicated reduced order methods
    - Quantum physics : a long history for the construction of tensor formats
    - Machine learning and statistical learning: a huge literature on reduced order models for high dimensional functions.
- A priori estimates
- Automatic selection of reduced order formats (bases or frames for sparse approximation, tensor formats for low-rank approximation).
- Well-conditioned formulations for (quasi-)optimal model reduction
- Samples-based constructions: How to sample given an approximation format ? How many samples ?
- Software engineering. Minimize interactions with existing codes.
- Goal-oriented model order reduction: variable of interest $s = \ell(u)$, rare event computation, sensitivity analysis, optimization, inverse problems, ...

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